Generic bounded solutions and maximal slope quotients of overconvergent F-isocrystals on curves

Nobuo TSUZUKI

Tohoku University

Périodes, motifs et équations différentielles:
entre arithmétique et géométrie
à l'occasion des 60^{++} ans d'Yves André

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Contents

§1. Minimal slope conjecture

k : perfect field, char $p > 0$

R : c.d.v.r., mixed char $(0, p)$ with $k = R/\mathbf{m}_K$

K : $= \mathrm{Frac}(R)$ with q-Frobenius σ ($q = p^a$ $(a > 0)$)

|-| : multi. valuation normalized by $|p| = p^{-1}$

X : smooth connected scheme sep. of f.t. $/k$

(over)convergent F-isocrystals (Berthelot, Crew, Kedlaya, $\cdots$)

$\mathcal{M}^{(\dagger)} = (\mathcal{M}^{(\dagger)}, \nabla, \Phi)$: coherent sheaf on a p-adic analytic lift of X

$$\begin{cases} \text{integrable connection, (over)convergent condition} \\ \text{Frobenius } \Phi : F^*\mathcal{M}^{(\dagger)} \overset{\cong}{\to} \mathcal{M}^{(\dagger)} \text{ horizontal w.r.t. } \nabla \end{cases}$$

$$\begin{pmatrix} \text{overconvergent} \\ F\text{-isocrystals on } X/K \end{pmatrix} \overset{\text{fully faithful}}{\subset} \begin{pmatrix} \text{convergent} \\ F\text{-isocrystals on } X/K \end{pmatrix}$$

$$\mathcal{M}^\dagger \quad \mapsto \quad \mathcal{M}$$

N.B. $X = \mathrm{Spec}\, k[\underline{x}]/I$: affine smooth

$$\Rightarrow \begin{cases} A^\dagger_{X,K} = (R[\underline{x}]^\dagger/I^\dagger) \otimes K = \{\sum_{\underline{n} \geq 0} a_{\underline{n}} \underline{x}^{\underline{n}} \mid \exists\, \eta > 1 \text{ s.t. } |a_{\underline{n}}|\eta^{|\underline{n}|} \to 0\}/I^\dagger_K & \cdots \quad \text{overconv.} \\ \widehat{A}_{X,K} = (\widehat{R[x]}/\widehat{I}) \otimes K = \{\sum_{\underline{n} \geq 0} a_{\underline{n}} \underline{x}^{\underline{n}} \mid |a_{\underline{n}}| \to 0\}/\widehat{I}_K & \cdots \quad \text{conv.} \end{cases}$$

Slopes and slope filtrations

Definition

$\mathcal{M} = (\mathcal{M}, \nabla, \Phi)$: a conv F-isoc on X/K

(1) $\mathcal{M}$ is isoclinic of slope 0 if $i_{\overline{a}}^{*}\mathcal{M} \cong (\widehat{K}^{\mathrm{ur}}, \sigma)^{\mathrm{rk}(\mathcal{M})}$ for $\forall\, i : \overline{a} \to X$.

(2) $\mathcal{M}$ is isoclinic of slope m/n if $(\mathcal{M}^{\otimes n}, q^{-m}\Phi^{\otimes n})$ is isoclinic of slope 0.

(3) Slope filtration of $\mathcal{M}$ is an increasing filtration

$$0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_r = \mathcal{M}$$

$\mathcal{M}_i/\mathcal{M}_{i-1}$: isoclinic of slope s_i satisfying $s_1 < s_2 < \cdots < s_r$.

Theorem (Katz, Crew)

$$\left(\begin{array}{c} \text{continuous} \\ p\text{-adic representations} \\ \text{of } \pi_1^{\mathrm{et}}(X, *) \end{array} \right) \cong \left(\begin{array}{c} \text{convergent} \\ F\text{-isocrystals on } X/K \\ \text{isoclinic of slope 0 (unit-root)} \end{array} \right)$$

Theorem (Katz, Kedlaya)

$\mathcal{M}$: conv F-isoc on X/K

$\Rightarrow\ \exists\, j : U \subset X$ dense open s.t. $j^{*}\mathcal{M}$ admits slope filtration.

Minimal slope conjecture

Conjecture (Kedlaya)

$\mathcal{M}^\dagger, \mathcal{N}^\dagger$: irred. o.c. F-isoc's on X/K admitting slope fil on $\mathcal{M}, \mathcal{N}$

$$\alpha_1 : \mathcal{M}_1 \overset{\cong}{\to} \mathcal{N}_1 \;\Rightarrow\; {}^\exists \alpha^\dagger : \mathcal{M}^\dagger \overset{\cong}{\to} \mathcal{N}^\dagger \text{ s.t. } \begin{tikzcd} \mathcal{M}_1 \arrow[r, "\alpha_1"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{N}_1 \arrow[d] \\ \mathcal{M} \arrow[r, "\alpha"'] & \mathcal{N} \end{tikzcd}$$

We will study the dual form:

Dual form of M.S.C.

$\mathcal{M}^\dagger, \mathcal{N}^\dagger$: irred. o.c. F-isoc's on X/K admitting slope fils on $\mathcal{M}, \mathcal{N}$

$$\mathcal{M} = \mathcal{M}^0 \supsetneq \mathcal{M}^1 \supsetneq \cdots \supsetneq \mathcal{M}^r = 0 \;\; (\mathrm{sl}(\mathcal{M}^{i-1}/\mathcal{M}^i) > \mathrm{sl}(\mathcal{M}^i/\mathcal{M}^{i+1})).$$

$$\beta^0 : \mathcal{N}/\mathcal{N}^1 \overset{\cong}{\to} \mathcal{M}/\mathcal{M}^1 \;\Rightarrow\; {}^\exists \beta^\dagger : \mathcal{N}^\dagger \overset{\cong}{\to} \mathcal{M}^\dagger \text{ s.t. } \begin{tikzcd} \mathcal{N} \arrow[r, "\beta"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M} \arrow[d] \\ \mathcal{N}/\mathcal{N}^1 \arrow[r, "\beta^0"'] & \mathcal{M}/\mathcal{M}^1 \end{tikzcd}$$

Results

Theorem (T. (curves), D'Addezio (in general))

The minimal slope conjecture is true.

○ Known results

- X : proper $\Rightarrow$ $\mathcal{M}^{\dagger} = \mathcal{M} = \mathcal{M}_1$ ($\because$ irred)
- [Ambrosi-D'Addezio] $\mathtt{rank}\mathcal{N}^{\dagger} = 1$, assuming only nontrivial α_1
- [T.] the case of curves, and in general over finite fields
  by studying b-solutions and max slope quotients
  + Chebotarev, Companion thm "$p \Rightarrow \ell$" (after Ambrosi-D'Addezio)
- [D'Addezio] †-hull ("dual" of our methods) + Lefschetz theorem
  $\Rightarrow$ Crew's parabolicity conjecture on monodromy groups of F-isoc

An application

nonisotrivial families of elliptic curves

$$\begin{tikzcd} Y_1 \arrow[dr] & & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

$\dim X > 0$

$U_i \subset X$: ordinary locus of Y_i/X

nonisotriv $\Rightarrow$ $U_i \neq \varnothing$

$\mathcal{M}_i^\dagger = \underline{H}^1_{\text{rig}}(Y_i/(X,\overline{X}))$

nonisotriv $\Rightarrow$ irred

$0 \to \mathcal{L}_i \to \mathcal{M}_i|_{U_i} \to \mathcal{N}_i \to 0$

slope 0 1

Tate module $\curvearrowleft \pi_1^{\text{et}}(U_i)$

$V_i = \mathbb{Q}_p \otimes \varprojlim Y_i[p^n](k(X)^{\text{sep}})$

$\mathcal{L}_i^\vee$: unit-root (slope 0)

$V_1 \cong V_2 \curvearrowleft \pi_1^{\text{et}}(U)$

for $\varnothing \neq \exists U \subset U_1 \cap U_2$

$\underset{\text{Katz-Crew}}{\Leftrightarrow}$

$\mathcal{L}_1|_U \cong \mathcal{L}_2|_U$

$\Updownarrow$ **M.S.C.** + F.F.

$Y_1 \overset{\text{isogeny}}{\sim} Y_2/X$ $\overset{\text{de Jong + Chai-Faltings}}{\Leftrightarrow}$ $\mathcal{M}_1^\dagger \cong \mathcal{M}_2^\dagger$

Theorem (Kedlaya)

$U \subset X$: open dense $\Rightarrow$ F-Isoc$^\dagger(X/K) \to F$-Isoc$^\dagger(U/K)$: fully faithful.

§2. Bounded solutions and maximal slope quotients

X : smooth connected affine curve over k

$\mathcal{M}^\dagger$: o.c. F-isocrystal on X/K

$M_\eta = \Gamma(]X[, \mathcal{M}) \otimes E_{X,K}$: generic (Φ, ∇)-module

Two quotients

$$\begin{array}{ccc} M_\eta/M_\eta^b & \twoheadrightarrow & M_\eta/M_\eta^1 \\ \text{max bounded quot} & & \text{max slope quot} \end{array}$$

Definition (Local version by Chiarellotto-T.)

$\mathcal{M}^\dagger$ is PBQ (pure of bounded quotient) $\Leftrightarrow$ surj is isom

Notation

X : a smooth connected affine curve over k

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "j"', "\cap"] & \mathcal{X} = \text{Spec}\, A_X \arrow[d, "\cap"] & \widehat{\mathcal{X}} = \text{Spf}\, \widehat{A}_X \arrow[d, "\cap"] & {]X[_{\widehat{\overline{\mathcal{X}}}}} = \text{Spm}\, \widehat{A}_{X,K} \arrow[l, "\text{sp}"'] \arrow[d, "\cap"] \\
\overline{X} \arrow[r] \arrow[d] & \overline{\mathcal{X}} \arrow[d, "\text{smooth}"] & \widehat{\overline{\mathcal{X}}} \arrow[d] & {]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}} = \overline{\mathcal{X}}_K^{\text{an}} \arrow[l, "\text{sp}"'] \arrow[d] \\
\text{Spec}\, k \arrow[r] & \text{Spec}\, R & \text{Spf}\, R & \text{Spm}\, K \arrow[l, "\text{sp}"']
\end{tikzcd}$$

- $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}} = \varinjlim_{V} \alpha_{V,*}\alpha_V^{-1}\mathcal{O}_{]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}}$ $\quad \alpha_V : V \subset]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}$ $\begin{cases} V \text{ : strict nbd of }]X[\text{ in }]\overline{X}[\\ \text{i.e., }]\overline{X}[= V \cup]\overline{X} \smallsetminus X[\text{ is admissible} \end{cases}$
- $A^{\dagger}_{X,K} = A^{\dagger}_X[1/p] = \Gamma(]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}, j^{\dagger}\mathcal{O}_{]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}})$ $\quad A^{\dagger}_X$: weak completion of A_X
- $\widehat{A}_{X,K} = \widehat{A}_X[1/p] = \Gamma(]X[_{\widehat{\overline{\mathcal{X}}}}, \mathcal{O}_{]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}})$ $\quad \widehat{A}_X$: p-adic completion of A_X
- $\mathcal{M}^{\dagger}$: o.c. F-isoc on X/K
- $M^{\dagger} = \Gamma(]\overline{X}[_{\widehat{\overline{\mathcal{X}}}}, \mathcal{M}^{\dagger})$: a fin. gen. projective $A^{\dagger}_{X,K}$-module with ∇, Φ
- $M = \Gamma(]X[_{\widehat{\overline{\mathcal{X}}}}, \mathcal{M})$: a fin. gen. projective $\widehat{A}_{X,K}$-module with ∇, Φ

Bounded quotients

$E_{X,K} := \widehat{A_{X,\mathbf{m}_K}}[1/p]$: p-adic lift of $k(X)$ with p-adic valuation $|\ |$

$t \in E_{X,K}$: a lift of certain local parameter of X

M_η : $E_{X,K}$-sp. of dim. n with $\nabla : M_\eta \to M_\eta \otimes_{E_{X,K}} \Omega_{E_{X,K}/K}$

$C_l \in M_n(E_{X,K})$ $(l \geq 0)$ s.t. $\frac{d^l}{dt^l}(e_1, \cdots, e_n) = (e_1, \cdots, e_n)C_l$

$Y = \sum_{l=0}^{\infty} \frac{C_l}{l!}(x-t)^l$: a matrix solution of M_η at generic point $x = t$

Definition (bounded modules)

(1) M_η is solvable $\Leftrightarrow$ $\left|\frac{C_l}{l!}\right| \eta^l \to 0$ (as $l \to \infty$) for $0 < {}^\forall \eta < 1$

(2) M_η is bounded $\Leftrightarrow$ $\sup_l \left|\frac{C_l}{l!}\right| < \infty$

Theorem (Dwork, Christol, Robba)

$$M_\eta : \text{solvable} \Rightarrow {}^{\exists}{}_1\, M_\eta^b \subsetneq M_\eta \text{ s.t. } \begin{cases} M_\eta/M_\eta^b \text{ is bounded} \\ \dim \mathrm{Sol}^b(M_\eta) = \dim M_\eta/M_\eta^b \end{cases}$$

$$\mathrm{Sol}^b(M_\eta) = \mathrm{Hom}_\nabla(M_\eta, E_{X,K}[[x-t]]_0)$$

Bounded/PBQ F-isoc $\mathcal{M}^{(\dagger)}$

Definition (Bounded/PBQ F-isoc)

(1) $\mathcal{M}^{(\dagger)}$ is bounded $\Leftrightarrow$ $M_\eta = M \otimes_{\widehat{A}_{X,K}} E_{X,K}$ is bounded

(2) $\mathcal{M}^{(\dagger)}$ is PBQ $\Leftrightarrow$ M_η / M_η^b has a unique slope $\Leftrightarrow$ $M_\eta^b = M_\eta^1$

Theorem (Chiarellotto-T. (local), T. (global))

$$\mathcal{M}^{(\dagger)} \text{ is bounded } \Leftrightarrow \begin{cases} \mathcal{M}^{(\dagger)} \text{ admits a slope fil s.t.} \\ \mathcal{M}^{(\dagger)} \cong \oplus_i (\mathcal{M}^{(\dagger)})^i / (\mathcal{M}^{(\dagger)})^{i+1} \end{cases}$$

Proposition

Given $\beta : \mathcal{N} \to \mathcal{M}$.

(1) $\mathcal{N}$ is PBQ & β is surj. $\Rightarrow$ g-sl$_{\text{max}}(\mathcal{N})$ = g-sl$_{\text{max}}(\mathcal{M})$ & $\mathcal{M}$ is PBQ

(2) $\mathcal{M}$: PBQ & $N_\eta / N_\eta^1 \overset{\beta^0}{\to} M_\eta / M_\eta^1$: surj $\Rightarrow$ β is surj.

Here g-sl$_{\text{max}}(\mathcal{M}) :=$ sl(M_η / M_η^1) : generic max. slope

PBQ filtration

Theorem (Chiarellotto-T. (local), T. (global))

$\mathcal{M}^\dagger$: o.c. F-isoc on X/K

$$\Rightarrow \quad \underset{\text{PBQ filtration}}{{}^{\exists_1}\ 0 = \mathcal{P}_0^\dagger \subsetneq \mathcal{P}_1^\dagger \subsetneq \cdots \subsetneq \mathcal{P}_r^\dagger = \mathcal{M}^\dagger} \ \text{s.t.} \begin{cases} \mathcal{P}_i^\dagger/\mathcal{P}_{i-1}^\dagger \text{ is PBQ} \\ \text{g-sl}_{\text{max}}(\mathcal{P}_i) > \text{g-sl}_{\text{max}}(\mathcal{P}_{i+1}) \end{cases}$$

- $\mathcal{P}_1^\dagger$ is called the max PBQ submodule of $\mathcal{M}^\dagger$

Theorem

$$\mathcal{M}^\dagger \text{ is irreducible} \Leftrightarrow \begin{cases} \mathcal{M}^\dagger \text{ is PBQ} \\ M^\dagger \to M_\eta/M_\eta^1 \text{ is injective} \\ M_\eta/M_\eta^1 \text{ is irreducible} \end{cases}$$

Generic inequality

Theorem

$\mathcal{M}$: isoclinic conv F-isoc on X/K, $\mathcal{N}^\dagger$: o.c. F-isoc on X/K

$$N^\dagger \subset M \curvearrowleft \Phi, \nabla \Rightarrow \begin{cases} \text{g-sl}_{\text{max}}(\mathcal{N}) = \text{sl}(\mathcal{M}) \\ \dim_{E_{X,K}} N_\eta / N_\eta^1 \leq \text{rk}\,\mathcal{M} \end{cases}$$

Prf of M.S.C. assuming inequality

Define an o.c. F-isoc $\mathcal{L}^\dagger$ by

$$L^\dagger = \text{Im}(M^\dagger \oplus N^\dagger \to M_\eta / M_\eta^1 \underset{\beta^0}{\overset{\cong}{\leftarrow}} N_\eta / N_\eta^1).$$

The inequality above implies

$$\dim M_\eta / M_\eta^1 \leq \dim L_\eta / L_\eta^1 \leq \dim M_\eta / M_\eta^1$$

so that $\mathcal{M}^\dagger \overset{=}{\to} \mathcal{L}^\dagger \overset{=}{\leftarrow} \mathcal{N}^\dagger$ by irreducibility (or PBQ properties of $\mathcal{L}^\dagger$). □

Relation with Dwork's conjecture on log-growth

Definition (log-growth $\lambda \geq 0$)

(1) M_η $(\mathcal{M}|_{]a[})$: log growth $\leq \lambda \Leftrightarrow \left|\frac{C_l}{l!}\right| = O(l^\lambda)$ $\left(\left|\frac{C_l(\widehat{a})}{l!}\right| = O(l^\lambda)\right)$

(2) Sol$_\lambda(M_*)$: solution space of log-growth $\leq \lambda$ $(* = \eta, a)$

Theorem (Dwork, André)

$\mathcal{M}$: conv. F-isoc on X/K

N.P. of $\{\text{Sol}_\lambda(M_\eta)\}_\lambda \prec$ N.P. of $\{\text{Sol}_\lambda(i_a^*\mathcal{M})\}_\lambda$

Theorem (Dwork, Chiarellotto-T., Ohkubo)

$\mathcal{M}$: PBQ conv. F-isoc on X/K

(1) generic/special log-growth fil of $\mathcal{M}$ = g/s slope fil of $\mathcal{M}^\vee$

(2) N.P. of $\{\text{Sol}_\lambda(M_\eta)\}_\lambda \prec$ N.P. of $\{\text{Sol}_\lambda(i_a^*\mathcal{M})\}_\lambda$

s.t. both endpoints coincide with each other.

§3. Inequality $\mathrm{rk}\,\mathcal{N}/\mathcal{N}^1 \leq \mathrm{rk}\,\mathcal{M}$

X : a smooth connected curve over $k = \overline{k}$

x : a lift of local parameter at $a \in X(k)$

$$\mathcal{R}_a = \lim_{\lambda \to 1^-} \Gamma([\lambda, 1), \mathcal{O}_{]\overline{X}[_{\widehat{\mathcal{X}}}}) = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \;\middle|\; \begin{array}{l} a_n \in K \\ |a_n|\eta^n \to 0\,(n \to \infty)\ (\forall \eta < 1) \\ \exists \lambda < 1 \text{ s.t. } |a_n|\lambda^n \to 0\,(n \to -\infty) \end{array} \right\}$$: Robba ring

$\mathcal{E}_a^\dagger = \{\sum a_n x^n \in \mathcal{R} \mid \sup_n |a_n| < \infty\}$: bounded Robba ring

$\mathcal{E}_a = \widehat{\mathcal{E}_a^\dagger}$: Amice ring, p-adic completion of $\mathcal{E}_a^\dagger$

$$\begin{tikzcd}
A^\dagger_{X,K} \arrow[r] \arrow[dr] \arrow[dd, phantom, "\cap"] & \mathcal{R}_a \arrow[d, phantom, "\cup"] & \\
 & \mathcal{E}_a^\dagger \arrow[d, phantom, "\cap"] & \cdots \text{ henselian d.v.f. with r.f. } k((x)) \\
\widehat{A}_{X,K} \arrow[r] & \mathcal{E}_a & \cdots \text{ complete d.v.f. with r.f. } k((x))
\end{tikzcd}$$

$\varphi \curvearrowright \mathcal{E}_a, \mathcal{E}_a^\dagger, \mathcal{R}_a$: a q-Frobenius s.t. $\varphi|_K = \sigma$ (e.g. $\varphi(x) = x^q$)

Definition

$M^\dagger$ is a φ-∇-module over $\mathcal{E}^\dagger$ if $\dim_{\mathcal{E}^\dagger} M^\dagger < \infty$ with

(i) $\nabla : M^\dagger \to M^\dagger \otimes_{\mathcal{E}^\dagger} \Omega_{\mathcal{E}^\dagger}$ K-connection

(ii) $\Phi : \varphi^* M^\dagger \overset{\cong}{\to} M^\dagger$ Frobenius, horizontal w.r.t. ∇.

Local inequality

$$\widetilde{\mathcal{E}} = \left\{ \sum_{n\in\mathbb{Q}} a_n x^n \;\middle|\; \begin{array}{l} a_n \in K,\ \text{sup}_n|a_n| < \infty, a_n \to 0\,(n \to -\infty) \\ \{n \,|\, |a_n| \geq \xi\} \text{ is well-ordered for } \forall \xi > 0. \end{array} \right\} \qquad \text{residue field } k((x^{\mathbb{Q}}))$$

$$\widetilde{\mathcal{E}}^\dagger = \{\textstyle\sum_n a_n x^n \in \widetilde{\mathcal{E}} \,|\, |a_n|\eta^n \to 0\,(n \to -\infty) \text{ for some } \eta < 1\}$$

Theorem (De Jong)

$N^\dagger$: a φ-module over $\mathcal{E}^\dagger$ with $\widetilde{N}^\dagger = \widetilde{\mathcal{E}}^\dagger \otimes_{\mathcal{E}^\dagger} N^\dagger$.

(1) $\exists_1$ opposite slope filtration $0 = \widetilde{N}_0^\dagger \subsetneq \widetilde{N}_1^\dagger \subsetneq \cdots \subsetneq \widetilde{N}_r^\dagger = \widetilde{N}^\dagger$ s.t.

$$\widetilde{\mathcal{E}} \otimes_{\widetilde{\mathcal{E}}^\dagger} \widetilde{N}_i^\dagger / \widetilde{N}_{i-1}^\dagger \cong \widetilde{\mathcal{E}} \otimes_{\mathcal{E}} N^{i-1}/N^i.$$

(2) $$\widetilde{N}^\dagger \subset \widetilde{\mathcal{E}} \curvearrowleft \varphi \Rightarrow \begin{cases} \text{sl}(N/N^1) = \text{sl}(\widetilde{N}_1^\dagger) = 0 \\ \text{dim}_{\mathcal{E}} N/N^1 = \text{dim}_{\widetilde{\mathcal{E}}^\dagger} \widetilde{N}_1^\dagger = 1 \end{cases}$$

Corollary

M : a φ-∇-module over $\mathcal{E}$ with a unique slope.
$N^\dagger$: a φ-∇-module over $\mathcal{E}^\dagger$

$$N^\dagger \subset M \curvearrowleft \varphi \Rightarrow \begin{cases} \text{sl}(N/N^1) = \text{sl}(M) \\ \text{dim}_{\mathcal{E}} N/N^1 \leq \text{dim}_{\mathcal{E}} M. \end{cases}$$

Local results

Theorem

$M^\dagger$: A φ-∇-module over $\mathcal{E}^\dagger$

(1) (Chiarellotto-T.) $\exists_1$ PBQ fil $\{P_i^\dagger\}$ of $M^\dagger$ s.t., if $M^\dagger$ arises from an o.c. F-isoc on X/K, then $\mathcal{E}_a^\dagger \otimes \mathcal{P}_i^\dagger = P_i^\dagger$ for $a \in X(k)$.

(2) $M^\dagger$: irreducible $\Leftrightarrow \begin{cases} M \text{ is PBQ} \\ M^\dagger \to M/M^1 \text{ is injective} \\ M/M^1 \text{ is irreducible} \end{cases}$

Theorem (Local minimal slope problem)

$M^\dagger$ and $N^\dagger$: irreducible φ-∇-modules over $\mathcal{E}^\dagger$

$\beta^0 : N/N^1 \overset{\cong}{\to} M/M^1 \;\Rightarrow\; {}^\exists \beta^\dagger : N^\dagger \overset{\cong}{\to} M^\dagger$ s.t.

$$\begin{tikzcd}
N \arrow[r, "\beta"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d] \\
N/N^1 \arrow[r, "\beta^0"'] & M/M^1
\end{tikzcd}$$

Proof of global inequality rk$\mathcal{N}/\mathcal{N}^1 \leq$ rk$\,\mathcal{M}$

1. We may assume $\mathcal{N}$ is PBQ. Then g-sl$_{\text{max}}(\mathcal{N}) =$ sl$(\mathcal{M})$.
2. We may assume $X = \mathbb{A}^1$, $\mathcal{N}$ admits the slope fil.

- Suppose rk$\mathcal{N}/\mathcal{N}^1 >$ rk$\,\mathcal{M}$.

$$\begin{tikzcd}
\mathcal{H} = \mathrm{Ker}(\mathcal{N} \to \mathcal{M}) \arrow[d, leftrightarrow, "\text{may not glue}"] & \supset & \overset{\text{max PBQ sub}}{\mathcal{L}} \arrow[d, leftrightarrow, "\text{glue in } \mathcal{N}^\dagger"] \\
H^\dagger_\infty = \mathrm{Ker}(N^\dagger \otimes_{A^\dagger_{X,K}} \mathcal{E}^\dagger_\infty \to M \otimes_{\widehat{A}_{X,K}} \mathcal{E}_\infty) & \supset & L^\dagger_\infty
\end{tikzcd}$$

$$\text{i.e.} \begin{cases} \mathcal{L}/\mathcal{L}^1 \cong \mathcal{H}/\mathcal{H}^1 \subsetneq \mathcal{N}/\mathcal{N}^1 \\ L^\dagger_\infty \otimes_{\mathcal{E}^\dagger_\infty} \mathcal{E}_\infty \overset{\cong}{\to} L_\infty \overset{\cong}{\leftarrow} L \otimes_{\widehat{A}_{X,K}} \mathcal{E}_\infty \ (\because \text{ global-local PBQ compatibility}) \end{cases}$$

- By gluing and full faithfulness from o.c. to conv.

$^\exists\mathcal{L}^\dagger$: o.c. F-isoc $\subset \mathcal{N}^\dagger$ s.t. $\begin{cases} \mathcal{L}^\dagger|_{]X[} \cong \mathcal{L} \\ L^\dagger \otimes \mathcal{E}^\dagger_\infty \cong L^\dagger_\infty \end{cases}$ contradicting $N^\dagger \subset M$

§4. Example of PBQ filtrations

p : odd prime number

$C = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}/\mathbb{F}_p$: z-line

χ : quadratic character w.r.t. $w^2 = z(z-1)$ on C

Proposition

$X : y^2 = x(x-1)(x-z)$ affine Legendre's family over C

$Y : y^2 = xz(x-1)(z-1)(x-z) \subset \overline{Y}$: singular K3-surface

(1) $\mathcal{L}^\dagger = \underline{H}^1_{\mathrm{rig}}(X/(C, \overline{C}))$: rk $= 2$, regular at $z = 0, 1, \infty$

(2) $H^1_{\mathrm{rig}}(C, \mathcal{L}^\dagger(\chi)) \cong H^1_{\mathrm{rig},c}(C, \mathcal{L}^\dagger(\chi)) \cong H^2_{\mathrm{rig},c}(Y)$ of $\dim = 2$

(3) $\mathrm{sl}_{\min}(H^2_{\mathrm{rig},c}(Y)) = 0$ if $p \equiv 1 \pmod 4$ and 1 if $p \equiv 3 \pmod 4$

$\because)$ $Z(\overline{Y}/\mathbb{F}_p, t) = \dfrac{1}{(1-t)(1-pt)^{20}(1-\rho_p\pi_p^2 t)(1-\overline{\rho}_p\overline{\pi}_p^2 t)(1-p^2 t)}$ (Stienstra-Beukers)

$L(X_{-1}/\mathbb{F}_p, t) = \det(1 - t\Phi; i^*_{-1}\mathcal{L}^\dagger) = (1 - \pi_p t)(1 - \overline{\pi}_p t)$, $\rho_p \in \mathbb{Z}[\sqrt{-1}]^\times$ □

Conclusion

$p \equiv 1 \pmod 4 \Rightarrow {}^\exists$ o.c. F-isoc $\mathcal{M}^\dagger$ on $C/\mathbb{Q}_p$ with nontriv. PBQ fil:

$$0 \to \mathcal{L}^\dagger(\chi) \to \mathcal{M}^\dagger \to j^\dagger\mathcal{O}_{]\overline{C}[} \otimes_{\mathbb{Q}_p} H^1_{\mathrm{rig}}(C, \mathcal{L}^\dagger(\chi))^{\mathrm{sl}=0} \to 0$$

I hope Yves and everyone will have happy and healthy life.

Thank you very much for your attention.