The Hecke Orbit conjecture

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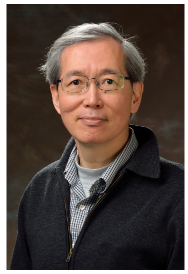
Periods, motives, and differential equations: between arithmetic and geometry.
On the occasion of Yves André's 60th^{++} birthday.

Consider a point $x = [(B, \mu)] \in \mathcal{A}_g$ in the moduli space of polarized abelian varieties (over some base field). The Hecke orbit of x, denoted by $\mathcal{H}(x)$, is the set of moduli points $y = [(C, \nu)]$ such that over some field Ω there exists a quasi-isogeny $(B, \mu)_\Omega \sim (C, \nu)_\Omega$.
Problem. Describe the closure $\mathcal{H}(x)^{\mathrm{Zar}}$.

We see a set of points, arithmetically defined in $\mathcal{A}_g$, and we ask for geometric properties of the Zariski closure. Where have we seen such a pattern before?

In 1989 Yves André formulated a stimulating conjecture. Soon his intuition and idea were generalized and after 30 years AO has been proved. Thank you Yves for your challenging problems and ideas.

It is a great pleasure that this ++-conference can be held. I wish you many happy returns Yves.



We report on joint work with Ching-Li Chai

(1) The Hecke orbit conjecture. For $x = [(B, \mu)] \in \mathcal{A}_g$ we write

$$\mathcal{H}(x) = \{[(B_1, \mu_1)] \mid \exists \otimes \Omega : \quad (B, \mu) \otimes \Omega \sim (B_1, \mu_1) \otimes \Omega\}.$$

(1.1) The HO problem is solved *in characteristic zero*:

$$\mathcal{H}(x) \textit{ is dense in } \mathcal{A}_g(\mathbb{C}).$$

Use complex uniformization and (even classical) density of $\mathrm{Sp}_{2g}(\mathbb{Q})$ in $\mathrm{Sp}_{2g}(\mathbb{R})$.

Notation. Fix g and choose an integer $n \geq 3$ prime to p. Write $d = (d_1, \cdots, d_m)$ for a set of elementary divisors (d_i divides d_{i+1}), and define

$$\mathcal{A}_g = \cup_d \ \mathcal{A}_{g,d,n} \otimes \mathbb{F}_p.$$

(1.2) **An example in positive characteristic.** *Let E be a supersingular elliptic curve over a field $K \supset \mathbb{F}_p$. Its Hecke orbits is finite in every irreducible component of $\mathcal{A}_1 \otimes \mathbb{F}_p$.*

In this case $\mathcal{H}(x)$ is not dense in any of the irreducible components of $\mathcal{A}_1 \otimes \mathbb{F}_p$.

Let E' be an ordinary elliptic curve over a field $K \supset \mathbb{F}_p$. Its Hecke orbits is dense in every irreducible component of $\mathcal{A}_1 \otimes \mathbb{F}_p$.

More generally, or any $x = [(B, \mu)] \in \mathcal{A}_g \otimes \mathbb{F}_p$, where $\xi = \mathcal{N}(B)$ is the Newton Polygon of B, its Hecke Orbit is contained in the NP stratum $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$. (Notation explained below.)
Hence: for $x \in \mathcal{W}_\xi(\mathcal{A}_{g,d,n}) \subsetneqq \mathcal{A}_{g,d,n}$ the Hecke orbit $\mathcal{H}(x)$ is not dense in $\mathcal{A}_{g,d,n}$.

Can we describe $\mathcal{H}(x)^{\mathrm{Zar}}$? In 1995 we see one answer and one conjecture:

(1.3) **Theorem HO**$-\rho$ (Ching-Li Chai, 1995).

> *For A* ordinary *$\mathcal{H}(x)$ is dense in $\mathcal{A}_g \otimes \mathbb{F}_p$.*

(1.4) **(HO) Conjecture** (FO, 1995) = **Theorem HO** (Chai + FO).

> *For $x = [(B, \mu)]$ the set $\mathcal{H}(x)$ is dense in $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$,*

the NP stratum with Newton Polygon $\xi = \mathcal{N}(B)$.

Some notation. We write k for an algebraically closed field. We write A for an abelian variety, and X for a p-divisible group.

We write $\mathcal{H}^{(p)}$, Hecke orbits involving isogenies of degree prime to p, and we write $\mathcal{H}_\ell$ with isogenies of degree a power of ℓ, where $\ell \neq p$ is a prime number.

We write $(\mathbf{HO})_\ell$ for the conjecture that for any $x \in \mathcal{W}_\xi(\mathcal{A}_g)$ with $\xi \neq \sigma$, where σ is the supersingular Newton Polygon, then the orbit $\mathcal{H}_\ell(x) \subset \mathcal{C}(x) \subset \mathcal{A}_g$ is dense in the central leaf $\mathcal{C}(x)$. (Notation explained below.)

The set $\mathcal{H}(x)$ is called the Hecke orbit of x; we say "the action of $\mathcal{H}$ on x", when considering the $\mathrm{Sp}_{2g}(\mathbb{A}_f)$-action.

We will see that the proof of (**HO**) discussed below reproves the result (1.3) **HO-**ρ by Chai, 1995.

(2) Survey of our proof of the Hecke Orbit conjecture.

In 1966 Grothendieck wrote to Mumford: "*I found it kind of astonishing that you should be obliged to dive so deep and so far in order to prove a theorem whose statement looks so simple-minded.*" This seems to apply also to our proof of (**HO**).

Survey of our proof.

- **Obvious reductions of the problem.** At first we observe some obvious reductions of the problem. As a corollary of the "almost product structure":

$$(\mathbf{HO})_\ell \Rightarrow (\mathbf{HO})$$

  and we can work over a finite field:

$$(\mathbf{HO})_\ell \text{ for } \mathcal{A}_{g,1,n} \otimes \overline{\mathbb{F}_p} \Rightarrow (\mathbf{HO})_\ell.$$

  We split the proof of $(\mathbf{HO})_\ell$ on $\mathcal{A}_{g,1,n} \otimes \overline{\mathbb{F}_p}$ into *two quite different aspects*:

- **The discrete part.** We discuss a proof of $(\mathbf{HO})_{\mathrm{discr}}$:

  for $x \in W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1,n})$ with $\xi \neq \sigma$ then W_ξ and $\mathcal{C}(x)$ are *geometrically irreducible.*

- **The continuous part.** We briefly sketch a proof of $(\mathbf{HO})_{\mathrm{cont}}$:

  for $x \in \mathcal{W}_\xi(\mathcal{A}_{g,d,n})(\overline{\mathbb{F}_p})$ with $\xi \neq \sigma$ then $\mathcal{H}_\ell(x) \subset \mathcal{C}(x)$ is dense with $\mathcal{C}(x) \subset \mathcal{W}_\xi(\mathcal{A}_{g,d,n})$.

Conclusion:
$(\mathbf{HO})_{\mathrm{discr}} + (\mathbf{HO})_{\mathrm{cont}} \Rightarrow (\mathbf{HO})$.

Pleasant surprise: although various questions about strata and leaves in the non-principally polarized cases can be difficult, for a proof of **(HO)** is suffices to prove the case of $\mathcal{A}_{g,1,n} \otimes \overline{\mathbb{F}_p}$.

(3) Strata and leaves in $\mathcal{A}_g$.
From now on write $\mathcal{A}_g = \cup_d \ \mathcal{A}_{g,d,n} \otimes \mathbb{F}_p$.
(Later we consider $\mathcal{A}_g = \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$.)

Many methods useful in characteristic zero are not available in positive characteristic. However there quite different concepts can be used.

The HO problem urged us to find the "underlying structure".

We discuss the following ingredients:

Newton Polygon strata (Manin, Grothendieck, Katz),
abelian varieties over finite fields (Tate),
Ekedahl-Oort strata, central leaves, isogeny leaves,
minimal abelian varieties, hypersymmetric abelian varieties and p-adic monodromy.

Basic, general approach:

find strata and leaves, intrinsically defined in $\mathcal{A}_g \otimes \mathbb{F}_p$,

their interplay, and give proofs by

degeneration to the "boundary".

Another basic ingredient: *hypersymmetric abelian varieties*, to be defined and discussed below.

We consider the proof (1995) by Chai of *density of a Heck orbit in the ordinary locus*:
consider the closure T of $\mathcal{H}_\ell(x)$ in a toroidal compactification of $\mathcal{A}_g$,
use "the cusp at infinity" $\infty \in T$, and
a delicate and careful study of Hecke-ℓ-stable subspaces of $T^{/\infty}$ proved
density of any Hecke orbit in the ordinary locus.

For prank(A) $= 0$ we cannot degenerate to the boundary of $\mathcal{A}_g$,
but, for every Newton Polygon ξ we can 'degenerate' to a smaller NP-stratum.
That will be our basic strategy.

We consider p-divisible groups. We write X^t for the Serre dual of X. Invariants of an abelian variety A will be given with the help $X = A[p^\infty]$.

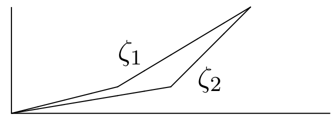
(3.1) Newton Polygons, NP-strata.
Manin and Dieudonné gave a classification of isogeny classes of p-divisible groups. Here are the basic ingredients. For coprime non-negative integers m and n there exists a simple p-divisible group $G_{m,n}$, defined over $\mathbb{F}_p$, of dimension m (we omit the precise definition), where $G^t_{m,n} = G_{n,m}$ has dimension n, and we define $\mathcal{N}(G_{m,n})$ as the line segment of (Frobenius) slope $m/(m+n)$ and length $h = m + n = \text{height}(G_{m,n})$; this Newton Polygon will be indicated by (m,n). Over any $k = \overline{k}$ any p-divisible group X admits an isogeny $X \sim \prod G_{m_i,n_i}$ (Dieudonné - Manin) and the *lower convex* Newton Polygon $\mathcal{N}(X) = \zeta = \cup_i \ \mathcal{N}(G_{m_i,n_i})$ classifies the k-isogeny class of X. Some properties:

Define a partial ordering on the set of Newton polygons. For Newton polygons ζ_1, ζ_2

$$\zeta_1 \preceq \zeta_2 \iff \text{ht}(\zeta_1) = \text{ht}(\zeta_2),\ \dim(\zeta_1) = \dim(\zeta_2) \text{ and no point of } \zeta_1 \text{ is strictly below } \zeta_2$$

We will say that "ζ_1 is *on or above* ζ_2" when $\zeta_1 \preceq \zeta_2$.

Illustration: $\zeta_1 \precneqq \zeta_2$



- For $X = A[p^\infty]$ the Newton Polygon $\xi = \mathcal{N}(X) =: \mathcal{N}(A)$ is symmetric (i.e. a slope s and $1-s$ appear with the same multiplicity). This follows from the duality theorem, which implies $A^t[p^\infty] \cong A[p^\infty]^t$.
- Newton Polygons "go up" under specialization (proved by Grothendieck).
- (Grothendieck-Katz) The locus

$$\mathcal{W}_\xi(\mathcal{A}_g) := \{[B,\mu] \mid \mathcal{N}(B) = \xi\} \subset \mathcal{A}_g$$

  is locally closed.

- Some special cases:

  ordinary: $\rho = g((1,0)+(0,1))$;

  almost ordinary: $a\rho = (g-1)(1,0)+(1,1)+(g-1)(0,1))$;

  supersingular $g(1,1)$.

- **Open problem.** Give a "functorial definition" of NP strata.
- The Hecke Orbit problem is solved for the supersingular stratum.
- The ordinary locus is dense in every irreducible component of $\mathcal{A}_g$ (Mumford, Norman-FO, 1980).
- For many Newton Polygons the locus $\mathcal{W}_\xi(\mathcal{A}_g)$ can have ireducible components of different dimensions; upper and lower bounds are precisely known.

Notation. We write $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1,n} \otimes \mathbb{F}_p)$ in the *principally polarized* case.

(3.2) EO-strata.
Here we only work in $\mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$. A *principally polarized* abelian variety defines $\varphi := (A[p], <>)$, a finite group scheme with a pairing, called a "polarized BT_1".
By a theorem of Kraft we know the number of such isomorphism classes for given g is *finite*.
Torsten Ekedahl and FO defined and studied

$$S_\varphi = \{[B,\mu] \mid \varphi \cong (B[p], <>) \otimes \Omega\} \subset \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$$

where Ω is some algebraically closed field. Some properties:

- (Ekedahl - Van der Geer) There exists a functorial definition of S_φ.
- $S_\varphi \subset \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$ is locally closed, and smooth over $\mathbb{F}_p$.
- Every S_φ is *quasi-affine*; a proof follows by the PhD-thesis of Laurent Moret-Bailly and a generalization of the "Raynaud trick".
- The EO-stratum S_φ has dimension zero (hence is a finite set of points) if and only if A is superspecial, i.e. $A_k \cong E^g$ for a supersingular elliptic curve E.

- We can define a partial ordering on the set (of isomorphism classes) of principally polarized BT_1 group schemes, and the EO stratification satisfies

$$S_{\varphi}^{\mathrm{Zar}} = \cup_{\psi \preceq \varphi} \;\; S_{\psi}.$$

- EO strata give access to specializing to smaller strata, and in this way irreducibility of $\mathcal{A}_{g,1,n} \otimes k$ (Faltings and Chai) has been reproved.

(3.3) Central leaves, isogeny leaves and the almost product structure.
Consider $x = [(A, \lambda)] \in \mathcal{A}_{g,d,n}$ and define the *central leaf* containing x:

$$\mathcal{C}(x) = \{[(B, \mu)] \in \mathcal{A}_{g,d,n} \mid (B, \mu)[p^{\infty}] \otimes \Omega \cong (A, \lambda)[p^{\infty}] \otimes \Omega\},$$

$$\mathcal{C}(x) \subset \mathcal{A}_{g,d,n}.$$

Some properties:

- For $\xi = \mathcal{N}(A)$ the central leaf is a closed subscheme $\mathcal{C}(x) \subset \mathcal{W}_{\xi}(\mathcal{A}_{g,d,n})$.
- We see a "pointwise definition", however *we have a functorial definition* using a new notion *"sustained p-divisible groups"*.
- $\mathcal{C}(x)$ is smooth over the base field.
- Some examples.

  For x ordinary or almost ordinary $\mathcal{C}(x) = \mathcal{W}_{\xi}(\mathcal{A}_{g,d,n})$.

  For x supersingular $\mathcal{C}(x)$ is finite and $\mathcal{H}^{(p)}(x)$ is finite in every irreducible component of $\mathcal{A}_g$.
- Central leaves are defined for any degree of the polarization.
- Note that prime-to-p Hecke actions "move points" inside a central leaf.

Consider Hecke actions using only isogenies with kernel a multiple extension of copies of α_p, thus defining $\mathcal{H}_{\alpha}$. In general $\mathcal{H}_{\alpha}(x)$ can have infinitely many irreducible components inside an irreducible component of $\mathcal{A}_g$. However for any x there are only finitely many irreducible components containing x; the union of these we denote by $I(x)$, called an *isogeny leaf.* Over a perfect base field we consider the reduced scheme structure on $I(x)$.

Note that every isogeny leaf in the ordinary stratum and in the almost ordinary stratum consist of one point.

Explanation. Note that the formal completion $I(x)^{/y}$ is isomorphic to the formal completion at y of a Rapoport-Zink space reduced modulo p with reduced structure.

- (Reminder.) Prime-to-p Hecke actions "move points" inside a central leaf.
- (Almost transversal.) Hecke-α actions "move points" inside an isogeny leaf.

(3.4) Theorem (The almost product structure, 2004). *Let ξ be a symmetric* NP *and let W be a geometrically irreducible component of $\mathcal{W}_\xi(\mathcal{A}_g \otimes k)$. There exist algebraic reduced schemes C and I over k, with I irreducible, and a surjective, finite morphism over k*

$$\Phi : C \times I \twoheadrightarrow W$$

such that

for any $c \in C$ the image $\Phi(\{x\} \times I) \subset W$ is an isogeny leaf, and

for any $z \in I$, the image $\Phi(C \times \{z\}) \subset W$ is a central leaf.

Comment. This is still correct inside $\mathcal{A}_{g,d,n}$ with moreover C irreducible.

Comment, warning. In general a central leaf and an isogeny leaf are not transversal at intersection points.

Corollary.

$$(\mathbf{HO})_\ell \Rightarrow (\mathbf{HO}).$$

Explanation. We see $\mathcal{H}_\alpha$ gives density in isogeny leaves, $\mathcal{H}_\ell$ "moves" in a central leaf; hence

density $\mathcal{H}_\ell(x) \subset \mathcal{C}(x)$ and the almost product structure
prove density $\mathcal{H}(x) \subset \mathcal{W}_\xi(\mathcal{A}_g)$.

Survey of strata and leaves.
We recall: fix g, and $n \geq 3$ prime to p, and write $\mathcal{A}_g = \cup_d \ \mathcal{A}_{g,d,n} \otimes \mathbb{F}_p$.

Newton Polygon strata. For a symmetric Newton Polgon define

$$\mathcal{W}_\xi(\mathcal{A}_g) := \{[B, \mu] \mid \mathcal{N}(B) = \xi\} \subset \mathcal{A}_g.$$

EO-strata. For $\varphi := (A[p], <>)$ define $S_\varphi \subset \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$,

$$S_\varphi = \{[B, \mu] \mid \varphi \cong (B[p], <>) \otimes \Omega\}$$

Central leaves. For $x = [(A, \lambda)] \in \mathcal{A}_{g,d,n}$ and define the *central leaf* containing x:

$$\mathcal{C}(x) = \{[(B, \mu)] \in \mathcal{A}_{g,d,n} \mid (B, \mu)[p^\infty] \otimes \Omega \cong (A, \lambda)p^\infty] \otimes \Omega\}.$$

Three p-adic invariants for polarized abelian varieties and the resulting stratification or foliation on $\mathcal{A}_g \otimes \overline{\mathbb{F}_p}$ are listed in the following table.

the isogeny class of X	ξ	NP	W_ξ
the isomorphism class of $(X[p], \lambda[p])$	φ	EO	$\mathcal{S}_\varphi$
the isomorphism class of (X, λ)	$[(X, \lambda)]$	CFol	$\mathcal{C}(x)$

(4) Two basic tools.

(4.1) Prime-to-p monodromy.

For an algebraic scheme S over a field K we write $\Pi_0(X)$ for the set of irreducible components of $S \otimes_K k$ for some algebraically closed field $k \supset K$.

Theorem (Chai, 2005). *Suppose an algebraic subscheme $W \subset \mathcal{A}_{g,d,n} \otimes k$ has no irreducible component contained in the supersingular locus $\mathcal{W}_\sigma(\mathcal{A}_{g,d,n} \otimes k)$. Suppose W is $\mathcal{H}^{(p)}$-stable and suppose $\mathcal{H}^{(p)}$ acts transitively on $\Pi_0(S)$. Then W is geometrically irreducible.*

(4.2) Hypersymmetric abelian varieties.

Definition (Chai-FO, 2006). An abelian variety A over $K \subset k := \overline{\mathbb{F}_p}$ is said to be *hypersymmetric* if

$$\mathrm{End}(A_k) \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}(A_k[p^\infty])$$

is an isomorphism.

Warning. Tate showed (1966) that for an abelian variety B over a finite field $K = \mathbb{F}_q$

$$\mathrm{End}(B) \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}(B[p^\infty]).$$

However, we will see there are many examples of an abelian variety $B/\mathbb{F}_q$ not hypersymmetric.

Some examples and properties.
(i) Elliptic curves are hypersymmetric.
(ii) For positive coprime integers $m > 0, n > 0$ there exists a hypersymmetric A with $\mathcal{N}(A) = (m,n) + (n,m)$. Products of hypersymmetric abelian varieties are hypersymmetric.
Conclusion. For every symmetric ξ there exists a hypersymmetric A with $\mathcal{N}(A) = \xi$.
(iii) Examples. An absolutely simple abelian surface ($g = 2$) of p-rank equal to $f = 1$ is not hypersymmetric. An absolutely simple ordinary abelian variety of dimension $g \geq 2$ is not hypersymmetric.
(iv) An absolutely simple abelian variety A is hypersymmetric is either a supersingular elliptic curve, or the following properties hold:
(iv-a) all slopes of $\xi := \mathcal{N}(A)$ have the same multiplicity, and
(iv-b) the greatest common divisor of the multiplicities of the simple parts of ξ is equal to 1.

(4.3) Generalized Serre-Tate coordinates.

For A_0 ordinary and $x_0 = [(A_0, \mu)]$ in characteristic p and in mixed characteristics on $\mathcal{A}_g \otimes \mathbb{Z}_p$ we have Serre-Tate coordinates on $(\mathcal{A}_g \otimes \mathbb{Z}_p)^{/x_0}$, canonical up to $\mathbb{Z}_p$-substitutions.

Can these be generalized to other NP-strata?

For x_0 not ordinary there is a generalization to $\mathcal{C}(x)^{/y}$ for every $y \in \mathcal{C}(x)$ to $\mathcal{A}_g \otimes \mathbb{F}_p$; however these cannot be extended to mixed characteristic.

For x_0 also non-almost-ordinary, these cannot be extended to the NP-stratum.

These generalized Serre-Tate structures on (completions of) central leaves will be of help in understanding the HO problem.

(4.4) Here is a *motivating example.* Choose $\xi = (2,1)+(1,2)$. Consider $x \in W_\xi \subset \mathcal{A}_{3,1,n}$. We know:

- $\dim(W_\xi) = 3$, $\dim(\mathcal{C}(X)) = 2$, isogeny leaves have dimension one;
- $\mathcal{C}(x)^{/y}$ canonically has the structure of a formal group, of dimension 2 and of Frobenius slope $2/3 - 1/3 = 1/3$.
- The geometry of $W_{(2,1)+(1,2)} \subset \mathcal{A}_{3,1,n} \otimes \mathbb{F}_p$ is well understood.
- For $\mathcal{N}(A) = \xi = (2,1)+(1,2)$ we

  either have $A[p^\infty]_k \cong G_{2,1} \times G_{1,2}$ and $a(A) = 2$, in this case $A[F] \cong (\alpha_p)^2$,

  or $a(A) = 1$, here $A[F] \not\cong (\alpha_p)^2$, and there exists an isomorphism $A[p^\infty]_k \cong (G_{2,1} \times G_{1,2})/\alpha_p$.
- If $a(A) = 1$ then $I(x)$ is an irreducible, rational curve.
- The $\mathcal{H}_\alpha$-orbit $H_\alpha(x) \subset W_\xi$ has a finite number of irreducible components if and only if A can be defined over a finite field.
- Although we seem to understand many details, still a proof for the Hecke Orbit problem even in this case is not easy.

For any ξ with two Frobenius slopes an analogous description is available. In the case of more than two slopes the description of $\mathcal{C}(x)^{/x_0}$ is more complicated, but well-understood (B. Moonen: cascades).

(5) A proof of $(\mathbf{HO})_{\rm discr}$.

Motivation, and a question.
We know the number of supersingular j-values, i.e. the number of geometric components of the supersingular locus for $g = 1$:

Deuring (1941), Eichler (1955) and Igusa (1958) give an interpretation of this as a class number.

For $g = 2$ the number of components of the supersingular locus was determined as a class number (Ibukiyama-Katsura-FO, 1986), and for arbitrary g this was done by K.-Z.Li - FO, 1998.
We see: inside $\mathcal{A}_{g,1,n}$

the supersingular NP-stratum has many irreducible components

(for p large). We should like to know (ir)reducibility of any NP-stratum for any g. It came as a surprise:

(5.1) Theorem (Chai-FO, 2011). *For any g and any $\xi \neq \sigma$ the* NP-*stratum* $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1,n})$ *is* geometrically irreducible.

For any $x \in W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1,n})$ the central leaf $\mathcal{C}(x)$ is geometrically irreducible.
Notation. Here we discuss a proof in case $\mathcal{A}_g = \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$. However this theorem can be generalized to $\mathcal{A}_{g,d,n}$.

(5.2) Only supersingular Hecke orbits are finite.
Inside any irreducible component T of $\mathcal{A}_g$

$$\mathcal{H}(x) \cap T \text{ is finite} \Leftrightarrow x \in \mathcal{W}_\sigma(\mathcal{A}_g) \Leftrightarrow \mathcal{H}_\ell(x) \cap T \text{ is finite.}$$

Using EO-strata we prove:

$$\text{For any } x \in \mathcal{A}_g, \text{ we have } \mathcal{H}_\ell(x)^{\mathrm{Zar}} \cap W_\sigma \neq \emptyset.$$

(5.3) Moduli of supersingular abelian varieties.
A long string of research (started by H. Hasse in 1936 with earlier roots by Gauss and Emil Artin) was completed by Ke-Zheng Li - FO in 1998
describing the *supersingular locus* $W_\sigma \subset \mathcal{A}_{g,1,n}$ and more generally $\mathcal{W}_\sigma(\mathcal{A}_g)$.
Here is one particular aspect we need here:

The action of $\mathcal{H}_\ell$ on $\Pi_0(W_\sigma)$ is transitive.

(5.4) Purity. Theorem (Aise Johan de Jong - FO, 2000). *Let $X \to S$ be a p-divisible group over an irreducible scheme S in characteristic p. Any jump of the Newton Polygon already takes place in codimension one.*
Comment. By Grothendieck-Katz we know jumps take place on a closed subscheme of S given by "many equations"; the proof that jumps already appear in codimension one is non-trivial.

(5.5) A conjecture by Grothendieck, 1970.
Grothendieck showed that "Newton Polygons go up under specialization" and Grothendieck, in his Montreal lecture 1970, posed a precise question about the converse. That can also be formulated for polarized p-divisible groups; that analogon does not hold for a an arbitrary degree of polarization. However for *principal polarizations* this analog of the Grothendieck conjecture does hold:

Theorem.

$$W_\xi^{\mathrm{Zar}} = \cup_{\xi' \preceq \xi} \ W_{\xi'} \subset \mathcal{A}_{g,1,n}.$$

(5.6) For a group scheme G over a field $K \supset \mathbb{F}_p$, and a perfect field $L \supset K$ we define

$$a(G) := \dim_L(\mathrm{Hom}(\alpha_p, G)).$$

(5.7) Theorem. *For a principally polarized (A_0, λ) there exists a deformation* (over an irreducible base scheme S) *to (A_η, λ) with $a(A_\eta) \leq 1$ and $\mathcal{N}(A_0) = \mathcal{N}(A_\eta)$.*

Comment. A proof of (5.5) and (5.7) uses a complicated combinatorial argument; it seems desirable to give a more conceptual proof.

Corollary. *For every symmetric ξ and every irreducible component $W \subset W_\xi \subset \mathcal{A}_{g,1,n}$ with generic point $\eta \in W$ we have $a(A_\eta) \leq 1$.* Another way of phrasing: $W(a \leq 1) \subset W$ is dense. This uses the methods developed in (5.4) Purity.

(5.8) Cayley-Hamilton. In general it is not easy to determine NP strata in a deformation space $\mathrm{Def}(A_0, \lambda)$. A non-commutative version of the Cayley-Hamilton theorem applied to a matrix displaying the Frobenius morphism leads in a particular case to a precise description: in the principally polarized case and with $a(A_0) = 1$ we know the structure of NP strata in $\mathrm{Def}(A_0, \lambda)$ very well.

Theorem. *Suppose $a(A_0) = 1$, and λ is a principal polarization; write $\mathcal{N}(A_0) = \xi \succeq \sigma$. In* $\mathrm{Def}(A_0, \lambda)$ *every Newton Polygon $\xi' \succeq \xi$ does appear, these* NP strata inside $\mathrm{Def}(A_0, \lambda)$ *are irreducible, formally smooth, they are transversal complete intersections, and they are nested* (under Zariski closure) *precisely as given by the partial ordering on Newton Polygons.*

Corollary. *For $\xi \preceq \xi'$ and for any irreducible $W \subset W_\xi \subset \mathcal{A}_{g,1,n} \otimes k$ there is precisely one irreducible component $W' \subset W_{\xi'} \subset \mathcal{A}_{g,1,n} \otimes k$ such that $W \subseteq (W')^{\mathrm{Zar}}$.*

Corollary. *For $\xi \preceq \xi'$ inclusion in the Zariski closure gives a well defined, surjective, Hecke-ℓ-equivariant map*

$$\Pi(W_\xi) \twoheadrightarrow \Pi(W_{\xi'}).$$

Note that Hecke-ℓ acts transitively on the set $\Pi(W_\sigma)$ of geometrically irreducible components of the supersingular stratum inside $\mathcal{A}_{g,1,n}$. Hence

Corollary. *Hecke-ℓ acts transitively on $\Pi(W_\xi)$ for every symmetric Newton Polygon ξ.*

Using prime-to-p monodromy, Chai 2005, this implies:

(5.8) Theorem *For every $\xi \neq \sigma$ the Newton Polygon stratum $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1,n})$ is absolutely irreducible.*

(5.9) Theorem *For $\xi \neq \sigma$ irreducibility of W_ξ implies irreducibility of $\mathcal{C}(x) \subset W_\xi$.*

Here is a way of deducing (5.9) from (5.8). Choose a principally polarized (A, λ) with *A hypersymmetric*. Using irreducibility of $W_\xi \subset \mathcal{A}_{g,1,n}$ it follows from the almost product structure that for for every irreducible component $C_i \subset \mathcal{C}(x) \otimes k$ there exists $(A_i, \lambda_i) \in (C_i \cap \mathcal{H}_\alpha(x))$. A careful study of endomorphism algebras and weak approximation shows:

Lemma. *For $x = [(A, \lambda)]$ and $[(A', \lambda')] = x' \in \mathcal{H}_\alpha(x) \cap \mathcal{C}(x)$ and A* hypersymmetric, *we also have $x' \in \mathcal{H}_\ell(x)$.*

Hence $\mathcal{H}_\ell$ operates transitively on $\Pi_0(C(x))$. Hence for non-supsingular $\mathcal{N}(A)$ we conclude $\mathcal{C}(x)$ is geometrically irreducible. QED $(\mathbf{HO})_{\mathrm{discr}}$

We used

$$\partial(W_\xi) := W_\xi^{\mathrm{Zar}} \backslash W_\xi = \cup_{\xi' \precneqq} W_{\xi'}$$

in the proof above. Can we follow an analogous path for central leaves? However we do not have enough information about $\partial(\mathcal{C}(x))$ and of deformation theory inside $\partial(\mathcal{C}(x)^{/z}$ (a hard unsolved problem) in order to prove irreducibility of central leaves directly along such a line.

(6) In order to find a proof of $(\mathbf{HO})_{\rm cont}$ we can try the following.

The proof (1995) by Chai of density of a Heck orbit in the ordinary locus:
consider the closure T of $\mathcal{H}_\ell(x)$ in a toroidal compactification of $\mathcal{A}_g$,
use "the cusp at infinity" $\infty \in T$, and
perform a delicate and careful study of Hecke-ℓ-stable subspaces of $T^{/\infty}$.

However, for any $x = [(A,\lambda)]$ of p-rank zero, $f(A) = 0$, with $\xi = \mathcal{N}(A)$ we know $\mathcal{W}_\xi(\mathcal{A}_g)^{\rm Zar} \subset \mathcal{A}_g$; the closure of $\mathcal{H}_\ell(x) \subset W_\xi$ does not contain moduli points of semi-abelian varieties, and degeneration to the boundary of $\mathcal{A}_g$ cannot be used.

But we can use another "degeneration":

(6.1) For any $x \in \mathcal{A}_{g,1,n}$

$$T := \mathcal{H}_\ell(x)^{\rm Zar} \cap W_\sigma \neq \emptyset.$$

Method?:
Choose $y \in T \cap W_\sigma$ and study Hecke-ℓ-stable subspaces of $T^{/y} \cap W_\xi$.
Failure. Despite many efforts Chai and I were unable to prove the desired Hecke-density along these lines.
We had to invent a *new method of proof* for the continuous part of **(HO)**.

We give a *very short sketch* of a proof of $(\mathbf{HO})_{\rm cont}$. In this part of the talk we give an outline, but we do not give any detailed proofs.

(6.2) Interlude, an example (due to Michael Larsen, 1995).
Let E be an ordinary elliptic curve over $\overline{\mathbb{F}_p}$, *with principal polarization λ_1 and consider $(B,\lambda) = (E,\lambda_1)^{\oplus g}$. Any $\mathcal{H}_\ell$-stable closed subspace $T \subset \mathcal{A}_{g,1,n}$ containing $x = [(B,\lambda)]$ is dense in $\mathcal{A}_{g,1,n}$.*
Note that this (B,λ) is hypersymmetric.

A proof involves:
the use of Serre-Tate coordinates on the ordinary locus, and
a careful study of the "group action" of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ on T.
Conclusion. Any $\mathcal{H}_\ell(x) \subset \mathcal{A}_g^{\rm ord}$ such that $\mathcal{H}_\ell(x)^{\rm Zar}$ contains the hypersymmetric $x = [(B,\lambda)]$ is dense in $\mathcal{A}_{g,1,n}$.

In order to find a proof of $(\mathbf{HO})_{\rm cont}$ we try find a hypersymmetric point in $\mathcal{H}_\ell(x)^{\rm Zar}$. Note that if A is not hypersymmetric, then (of course) $\mathcal{H}([(A,\mu)])$ does not contain a hypersymmetric point. However, it may happen that $\mathcal{H}([(A,\mu)])^{\rm Zar}$ does contain a hypersymmetric point.
Ching-Li Chai found a wonderful method, *the Hilbert trick*, that finally gave us access to this problem. We only give a rough sketch of these arguments here.

By a result of Tate (1966) we know that any abelian variety $A/\mathbb{F}_q$ of dimension g has "enough real multiplications", i.e. there is a Hilbert Modular variety M, associated with a totally real $E_1 \times \cdots \times E_m = E/\mathbb{Q}$ of rank g, consider its image in $\mathcal{A}_g$, such that $[(A,\mu)] \in M \subset \mathcal{A}_g$. On a HM variety the study of strata and leaves is easier. With a lot of effort, and joint work with Chia-Fu Yu we were able to show:

(6.3) (IV) Theorem (Yu-Chai-FO, 2003-2006-2020). *The Hecke orbit conjecture holds for Hilbert Modular varieties.*
The main difficulty turned out to find a way to find a delicate generalization of EO-strata in HM varieties in the case of ramification in the totally real algebra $E/\mathbb{Q}$.

(6.4) Corollary. *For any* $x = [(A, \mu)]$*, with* $\xi = \mathcal{N}(A) \neq \sigma$ *and for any* $\mathcal{H}_\ell$*-invariant closed subspace* $T \subset \mathcal{C}(x)$ *there exists a hypersymmetric point* $y \in T$*.*

Using this a proof of $(\mathbf{HO})_{\rm cont}$ follows:

(6.5) Corollary. *For any* $\xi \neq \sigma$ *we have* $\mathcal{H}_\ell(x)^{\rm Zar} = \mathcal{C}(x)$*.*

(6.6). **Conclusion. Theorem (HO).** *For any* $x = [(A, \mu)]$*, with* $\xi := \mathcal{N}(A)$*, the Hecke Orbit* $\mathcal{H}(x)$ *is dense in* $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$*.*

Postscript. Under Construction. It might be that a more direct proof of **(HO)** is possible:
study the structure of formal completions at closed points of a central leaf,
rigidity for Tate-linear formal varieties and
a p-adic monodromy argument.
At the moment no claim is possible that this will work, but there seems hope for a proof along these lines.

I wish all of you a pleasant conference, and wish Yves many more and healthy years in our beautiful mathematics. Thank you for your attention.

7 Some open problems.

(7.1) Give a scheme-theoretic definition of NP strata.

Note that all components of $\mathcal{A}_g \otimes \mathbb{F}_p$ are generically smooth; does the analogous result hold for other NP-strata?

Remark: there is a satisfactory scheme-theoretic definition for central leaves by the notion of "sustained p-divisible groups";
There is a satisfactory scheme-theoretic definition for EO strata, by Ekedahl - Van der Geer.

(7.2) For any ξ and any $x \in W_\xi$ describe $\partial(\mathcal{C}(x)) := (\mathcal{C}(x))^{\mathrm{Zar}} - \mathcal{C}(x)$.

Remark: for "central streams" this has been done because these are EO strata. Do other central leaves give the same answer?

(7.3) Is there a conjecture in positive characteristic parallel to the AO conjecture in characteristic zero, i.e. characterizing "Tate-linear" subvarieties of $\mathcal{A}_g \otimes \mathbb{F}_p$?

(7.4) What can be said about density of Hecke orbits in other moduli spaces, and in other Shimura varieties?

Ching-Li Chai and Frans Oort