Periods, Motives and Differential Equations: between Arithmetic and Geometry

on the occasion of Yves André's 60th birthday

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New perspectives on de Rham cohomology, after Bhatt-Lurie, Drinfeld, et al.

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0. Introduction

k perfect field of char. $p > 0$; X/k smooth

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d] & X \arrow[l, "F"'] \arrow[dl] \\ \mathrm{Spec}(k) & \mathrm{Spec}(k) \arrow[l, "\varphi:=F_k"'] \end{tikzcd}$$

Cartier isomorphism

$$C^{-1} : \Omega^i_{X'/k} \xrightarrow{\sim} H^i(F_*\Omega^\bullet_{X/k})$$

By [DI] a smooth lifting $\widetilde{X}/W_2(k)$ of X gives a decomposition in $D(X', \mathcal{O}_{X'})$

(0.1) $$\oplus_{0 \leqslant i < p} \Omega^i_{X'/k}[-i] \xrightarrow{\sim} \tau^{<p} F_*\Omega^\bullet_{X/k}$$

inducing C^{-1} on H^i.

Recall the main steps of the proof:

(1) $(\tau^{\geqslant -1} L\Omega^1_{X'/W(k)})[-1] \xrightarrow{\sim} \tau^{\leqslant 1} F_* \Omega^\bullet_{X/k}$ in $D(X', \mathcal{O}_{Y'})$
(use local liftings of Frobenius)

(2) $\widetilde{X}/W_2(k)$ lifting X gives splitting of $L\Omega^1_{X/W_2(k)}$
(elementary cotangent complex theory)

(3) from $\tau^{\leqslant 1}$ to $\tau^{<p}$: multiplicativity

Goal today: give an idea of the proof of the following theorem of Bhatt-Lurie (refining an unpublished, independent one of Drinfeld):

Theorem 1 [BL1, 5.16]. Let X/k be smooth. A smooth lifting $\widetilde{X}$ of X to $W_2(k)$ determines a commutative algebra object[1]

$$\Omega^{\not{D}}_{X/k} \in D(X, \mathcal{O}_X),$$

depending functorially on $\widetilde{X}$, which is a perfect complex, equipped with an endomorphism Θ and an isomorphism in $D(X', \mathcal{O}_{X'})$

$$\varepsilon : \varphi^* \Omega^{\not{D}}_{X/k} \xrightarrow{\sim} F_* \Omega^{\bullet}_{X/k}, \tag{0.2}$$

with the following properties:

(i) $H^i(\Omega^{\not{D}}_{X/k}) \xrightarrow{\sim} \Omega^i_{X/k}$ canonical, multiplicative;

(ii) Θ is a derivation, and acts by $-i$ on H^i;

(iii) ε is multiplicative and induces the Cartier isomorphism $C^{-1} : \Omega^i_{X'/k} \xrightarrow{\sim} H^i(F_* \Omega^{\bullet}_{X/k})$ on H^i.

[1] From now on, derived categories are taken in the derived ∞-categorical sense.

The complex $\Omega^{\not{D}}_{X/k}$ is called the diffracted Hodge complex of X (relative to $\widetilde{X}$), and Θ the Sen operator.

Let $d = \dim(X)$. As, by (ii) and (iii), $\prod_{0 \leqslant i \leqslant d}(\Theta + i)$ is nilpotent, we get a decomposition into generalized eigenspaces:

Corollary. Under the assumptions of th. 1, there is a canonically defined endomorphism Θ of $F_* \Omega^\bullet_{X/k}$, and a canonical decomposition

$$F_* \Omega^\bullet_{X/k} = \oplus_{i \in \mathbf{Z}/p\mathbf{Z}} (F_* \Omega^\bullet_{X/k})_i$$

stable under Θ, such that, for all $i \in \mathbf{Z}/p\mathbf{Z}$,

$$\Theta | (F_* \Omega^\bullet_{X/k})_i = -i + \Theta_i,$$

where Θ_i is a nilpotent endomorphism of $(F_* \Omega^\bullet_{X/k})_i$.

In particular, for any $a \in \mathbf{Z}$,

(0.2) $$\varepsilon : \varphi^* \Omega^{\not{D}}_{X/k} \xrightarrow{\sim} F_* \Omega^{\bullet}_{X/k},$$

induces a canonical decomposition

(0.3) $$\oplus_{a \leqslant i < a+p} \Omega^{i}_{X'/k}[-i] \xrightarrow{\sim} \tau^{[a,a+p-1]} F_* \Omega^{\bullet}_{X/k},$$

a refinement of (0.1) and of a result of Achinger-Suh [A].

Remarks. (1) It can be shown [LM] that, for $a = 0$, the decomposition (0.3) coincides with that of [DI] (see end of section 6).

(2) Petrov [P] has constructed an example of a smooth X/k, lifted to $W(k)$, such that $\Theta_0 \neq 0$.

(3) Suppose X admits a smooth formal lift $Z/W(k)$ with $Z \otimes W_2(k) = \widetilde{X}$. Then $\Omega^{\not{D}}_{X/k} = \Omega^{\not{D}}_{Z/W(k)} \otimes^{L} k$, where $\Omega^{\not{D}}_{Z/W(k)}$ is a certain perfect complex of $\mathcal{O}_Z$-modules, again called a diffracted Hodge complex, and Θ is induced by an endomorphism Θ of $\Omega^{\not{D}}_{Z/W(k)}$, inducing $-i$ on $H^i(\Omega^{\not{D}}_{Z/W(k)}) = \Omega^{i}_{Z/W(k)}$.

1. Strategy

For simplicity and convenience of references to [BL] we'll work with liftings $/W(k)$ rather than $/W_2(k)$ (adaptation to $/W_2(k)$ easy but technical, see [BL1, 5.16] and remarks at the end of section 5).

Change notation. Fix (formal smooth) $X/W(k)$ lifting $Y := X \otimes_{W(k)} k$.

Main steps

(a) First assume X affine, $X = \mathrm{Spf}(R)$. Using Hodge-Tate cohomology of X with respect to all (bounded) prisms over $\mathrm{Spf}(W(k))$ (forming the so-called absolute prismatic site $\mathrm{Spf}(W(k))_{\Delta}$ of $\mathrm{Spf}(W(k))$, construct an absolute Hodge-Tate crystal $\mathcal{C}_R$ in p-complete complexes over $\mathrm{Spf}(W(k))_{\Delta}$.

(b) A new input (Bhatt-Lurie, Drinfeld): the Cartier-Witt stack WCart and the Hodge-Tate stack over $\mathrm{Spf}(W(k))$:

$$\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W(k))} \hookrightarrow \mathrm{WCart}$$

(an effective Cartier divisor in the Cartier-Witt stack). The universal property of Witt rings with respect to δ-structures implies that WCart plays the role of an attractor for absolute prisms over $\mathrm{Spf}(W(k))$.

Using this, interpret (p-complete) absolute Hodge-Tate crystals on $\mathrm{Spf}(W(k))$ as (p-complete) complexes on the Hodge-Tate stack.

Then realize the Hodge-Tate stack as the classifying stack BG of a certain commutative affine group scheme G over $\mathrm{Spf}(W(k))$ containing μ_p, and whose quotient G/μ_p is isomorphic to $\mathbf{G}_a^\sharp$, the PD-hull of $\mathbf{G}_a$ at the origin.

(c) Interpret (p-complete) complexes on BG as (p-complete) complexes K of $W(k)$-modules, endowed with a certain endomorphism Θ, called the Sen operator, such that $\Theta^p - \Theta$ is nilpotent on each $H^i(K \otimes^L k)$. In particular, the Hodge-Tate crystal $\mathcal{C}_R$ above can be described by a p-complete object

$$\Omega^{\not{D}}_{R/W(k)}$$

of $D(W(k))$, endowed with a Sen operator Θ. Using the Hodge-Tate comparison theorem, promote $\Omega^{\not{D}}_{R/W(k)}$ to a perfect complex in $D(R)$, endowed with a multiplicative, increasing filtration $\mathrm{Fil}^{\mathrm{conj}}_\bullet$, with $\mathrm{gr}_i \xrightarrow{\sim} \Omega^i_{R/W(k)}[-i]$, and Θ acting on gr_i by $-i$

(d) For a general formal smooth $f : X \to \mathrm{Spf}(W(k))$, pasting the $\Omega^{\not D}_{R/W(k)}$ for the various affine opens $U = \mathrm{Spf}(R)$ of X gives a filtered perfect complex of $\mathcal{O}_X$-modules

$$\Omega^{\not D}_{X/W(k)}$$

called the diffracted Hodge complex of $X/W(k)$ (a twisted form of the Hodge complex $\oplus \Omega^i_{X/W(k)}[-i]$), with $\mathrm{gr}_i \xrightarrow{\sim} \Omega^i_{X/W(k)}[-i]$, and equipped with a Sen operator Θ, acting by $-i$ on $H^i(\Omega^{\not D}_{X/W(k)})$.

(e) Show that the object of $D(X, \mathcal{O}_X)$ underlying $\Omega^{\not{D}}_{X/W(k)}$ is the relative Hodge-Tate cohomology complex $\overline{\mathbb{\Delta}}_{X/A}$ of X over a special prism (A, I) with $A/I = W(k)$, deduced from the q-de Rham prism (by taking invariants under $\mathbf{F}_p^*$ and base changing to $W(k)$).

Finally, using this and the prismatic de Rham comparison theorem for $\overline{\mathbb{\Delta}}_{X/A}$, construct the desired isomorphism

$$\Omega^{\not{D}}_{Y'/k} \xrightarrow{\sim} F_* \Omega^{\bullet}_{Y/k} \tag{0.2}$$

inducing C^{-1} on H^i, where

$$\Omega^{\not{D}}_{Y/k} := \Omega^{\not{D}}_{X/W(k)} \otimes_{W(k)} k.$$

2. Prismatic and Hodge-Tate cohomology complexes

Recall: Prism: (A, I, δ):

• $\delta : A \to A$: a delta structure; $x \mapsto \varphi(x) := x^p + p\delta(x)$ lifts Frobenius

$(\delta \leftrightarrow (\text{section of } W_2(A) \to A) \leftrightarrow (\text{lift of } F : A/^L p \to A/^L p))$

• $I \subset A$: a Cartier divisor; A: derived (p, I)-complete (i.e., $A \xrightarrow{\sim} R\varprojlim A_{(t_1 \mapsto p^n, t_2 \mapsto d^n)} \otimes^L_{\mathbf{Z}[t_1,t_2]} \mathbf{Z}$ if $I = (d)$)

• $p \in I + \varphi(I)A$ ($\Leftrightarrow$ I locally generated by distinguished d, i.e. $\delta(d) \in A^*$)

Map of prisms: $(A, I) \to (B, J)$; $J = IB$ automatic

Examples: $(\mathbf{Z}_p, (p), \varphi(x) = x)$ (the crystalline prism);
$(\mathbf{Z}_p[[u]], (u - p), \varphi(u) = u^p)$ (a Breuil-Kisin prism);

$(\mathbf{Z}_p[[q-1]], ([p]_q := 1 + q + \cdots + q^{p-1}), \varphi(q) = q^p)$ (the q-de Rham prism)

(A, I) bounded: $A[p^\infty] = A[p^N]$, some N ($\Rightarrow A$ classically (p, I)-complete)

(Relative) prismatic site [BS, 4.1]:

For (A, I) bounded, $X/(A/I)$ smooth formal, prismatic site $(X/A)_{\mathbb{\Delta}}$:

Objects:

$$\begin{tikzcd} \mathrm{Spf}(B/IB) \arrow[r] \arrow[d] & \mathrm{Spf}(B) \arrow[dd] \\ X \arrow[d] & \\ \mathrm{Spf}(A/I) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd},$$

with (B, IB) bounded.

Maps: obvious

Covers: $(B, IB) \to (C, IC)$ faithfully flat (i.e., C (p, IB)-completely faithfully flat over B, [BS, 1.2])

Structure sheaf $\mathcal{O}_{\Delta}$: $(X \leftarrow \mathrm{Spf}(B/I) \to \mathrm{Spf}(B)) \mapsto B$;

Hodge-Tate sheaf $\overline{\mathcal{O}}$: $(-) \mapsto B/IB$; both with φ-actions

Let (A, I) a bounded prism, set $\overline{A} := A/I$. Fix $X/\overline{A}$ smooth formal.

Prismatic cohomology complex [BS, 4.2]:

$$\Delta_{X/A} := R\nu_* \mathcal{O}_{\Delta} \in D(X_{et}, A)$$

where ν is the canonical map of topoi

$$\nu : \widetilde{(X/A)}_{\Delta} \to \widetilde{X_{et}}.$$

$$R\Gamma_{\Delta}(X/A) := R\Gamma((X/A)_{\Delta}, \mathcal{O}) = R\Gamma(X_{et}, R\nu_* \mathcal{O}_{\Delta}) \in D(A).$$

Hodge-Tate cohomology complex [BS, 4.2]

$$\overline{\Delta}_{X/A} := R\nu_* \overline{\mathcal{O}}_{\Delta} = \Delta_{X/A} \otimes^L_A \overline{A} \in D(X_{et}, \mathcal{O}_X).$$

The Hodge-Tate comparison theorem [BS 4.11] gives an $\mathcal{O}_X$-linear isomorphism

$$(HT) \qquad \Omega^i_{X/\overline{A}} \otimes (I/I^2)^{\otimes -i} \xrightarrow{\sim} H^i(\overline{\Delta}_{X/A}),$$

hence $\overline{\Delta}_{X/A}$ is a perfect complex of $\mathcal{O}_X$-modules (with φ-action).

On the other hand, the de Rham comparison theorem [BS 15.4] gives a φ-linear isomorphism (in $D(X, \overline{A})$)

$$(dR) \qquad \varphi_A^* \Delta_{X/A} \otimes^L \overline{A} \; (\xrightarrow{\sim} \varphi_A^* \overline{\Delta}_{X/A}) \xrightarrow{\sim} \Omega^\bullet_{X/\overline{A}}.$$

Absolute prismatic site [BL, 4.4.27]:

For a p-adic formal scheme $T/\mathbf{Z}_p$, define the absolute prismatic site of T,

$$T_{\Delta},$$

as the category of maps

$$T \xleftarrow{a} \mathrm{Spf}(\overline{A}) \to \mathrm{Spf}(A)$$

where $(A, I), \overline{A} = A/I$, runs through all bounded prisms, with the the obvious maps, and the topology given by maps with $(A, I) \to (B, IB)$ faithfully flat.

Structural sheaves: $\mathcal{O}_{\Delta}$ (or $\mathcal{O}$): $(A, I) \mapsto A$, $\overline{\mathcal{O}}_{\Delta}$ (or $\overline{\mathcal{O}}$): $(A, I) \mapsto A/I$

Example: $\mathrm{Spf}(\mathbf{Z}_p)_{\Delta}$ is the category of all bounded prisms.

Definition. $\widehat{D}(A)$:= full subcategory of $D(A)$ consisting of (p, I)-complete objects.

Definitions. (1) A *p-complete prismatic crystal* on $T_{\mathbb{\Delta}}$ is an object E of $D(T_{\mathbb{\Delta}}, \mathcal{O})$ such that, for all $A = (A, I, a) \in T_{\mathbb{\Delta}}$, $E(A) \in \widehat{D}(A)$, and for all maps $(A, I, a) \to (B, IB, b)$, the induced map

$$B \widehat{\otimes}_A^L E(A) \to E(B)$$

is an isomorphism. Denote by

$$\widehat{D}_{\mathrm{crys}}(T_{\mathbb{\Delta}}, \mathcal{O})$$

the full subcategory of $D(T_{\mathbb{\Delta}}, \mathcal{O})$ consisting of p-complete prismatic crystals.

(2) A p-complete Hodge-Tate crystal on $T_{\mathbb{\Delta}}$ is an object E of $D(T_{\mathbb{\Delta}}, \overline{\mathcal{O}})$ such that, for all $A = (A, I, a) \in T_{\mathbb{\Delta}}$, $E(\overline{A}) \in \widehat{D}(\overline{A})$, and for all maps $(A, I, a) \to (B, IB, b)$, the induced map

$$\overline{B} \widehat{\otimes}^{L}_{\overline{A}} E(\overline{A}) \to E(\overline{B})$$

is an isomorphism. Denote by

$$\widehat{D}_{\mathrm{crys}}(T_{\mathbb{\Delta}}, \overline{\mathcal{O}})$$

the full subcategory of $D(T_{\mathbb{\Delta}}, \overline{\mathcal{O}})$ consisting of p-complete Hodge-Tate crystals.

Examples. Let $f : X \to \mathrm{Spf}(W(k))$ be formal smooth. Then:

(1)

$$Rf_*\mathcal{O}_{\Delta},$$

$$(A, I, a) \mapsto (Rf_*\mathcal{O}_{\Delta})_{(A,I)} = R\Gamma_{\Delta}(X_{\overline{A}}/A) := R\Gamma((X_{\overline{A}}/A)_{\Delta}, \mathcal{O}_{\Delta}) \in \widehat{D}(A)$$

belongs to $\widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(W(k))_{\Delta}, \mathcal{O})$

(2)

$$Rf_*\overline{\mathcal{O}}_{\Delta},$$

$$(A, I, a) \mapsto (Rf_*\overline{\mathcal{O}}_{\Delta})_{(A,I)} = R\Gamma_{\overline{\Delta}}(X_{\overline{A}}/A) := R\Gamma((X_{\overline{A}}/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \in \widehat{D}(\overline{A})$$

belongs to $\widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(W(k))_{\Delta}, \overline{\mathcal{O}})$

3. A strange attractor: the Cartier-Witt stack

A priori, the category $\mathrm{Spf}(\mathbf{Z}_p)_{\Delta}$ of all bounded prisms looks chaotic. However, these prisms are in a kind of magnetic field: they are all attracted by a single big formal stack, the Cartier-Witt stack. The local symmetries of this stack, and of its companion the Hodge-Tate stack, yield the hidden structure on $\Omega^{\bullet}_{Y/k}$ described above.

Definition [BL, 3.1.1]. A generalized Cartier divisor on a scheme X is a pair $(\mathcal{I}, \alpha)$, where $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module and $\alpha : \mathcal{I} \to \mathcal{O}_X$ a morphism of $\mathcal{O}_X$-modules. When $X = \mathrm{Spec}(A)$, we identify generalized Cartier divisors on X with pairs (I, α), where I is an invertible A-module and $\alpha : I \to A$ an A-linear map. Morphisms are defined in the obvious way.

Remarks

1. This notion, under the name of "divisor", was introduced by Deligne in 1988[2]. A similar notion was independently devised by Faltings at about the same time [F].

2. A generalized Cartier divisor on X generates (corresponds to) a log structure M_X on X called a Deligne-Faltings log structure of rank 1. In the early 2000's Lafforgue observed that such an M_X corresponds to a morphism $X \to [\mathbf{A}^1/\mathbf{G}_m]$. That triggered Olsson's work [Ol].

3. A generalized Cartier divisor (I, α) on A defines a quasi-ideal in A in the sense of Drinfeld [Dr], i.e. a differential graded algebra $(I \stackrel{\alpha}{\to} A)$ concentrated in degree 0 and -1, hence an animated ring $[I \stackrel{\alpha}{\to} A]$ (object of the derived category of simplicial (commutative) rings).

[2] Letter to L. Illusie, June 1, 1988

The Cartier-Witt stack (Drinfeld's Σ) is the formal stack over $\mathbf{Z}_p$

$$\mathrm{WCart} := [\mathrm{WCart}_0 / W^\times]$$

where:

W:= (p-typical) Witt scheme over $\mathbf{Z}_p$

WCart_0:= formal completion of W along locally closed subscheme defined by $p = x_0 = 0$, $x_1 \neq 0$, the formal scheme of primitive Witt vectors:

$$\mathrm{WCart}_0 = \mathrm{Spf}(A^0)$$

$$A^0 := \mathbf{Z}_p[[x_0]][x_1, x_1^{-1}, x_2, x_3, \cdots])\hat{}$$

where hat means p-completion (and the ring structure on A^0 is given by the Witt polynomials).

For a p-nilpotent ring R, $\mathrm{WCart}_0(R)$ is the set of $a = (a_0, a_1, \cdots) \in W(R)$ with a_0 nilpotent and a_1 invertible ($\Leftrightarrow$ $\delta(a)$ invertible[3]).

$W^\times \subset W$:= $\mathbf{Z}_p$-group scheme of units in W, acting on WCart_0 by multiplication.

[3] NB. $Fa = (a_0^p + pa_1, \cdots)$, $a^p = (a_0^p, \cdots)$, $\delta(a) = (a_1, \cdots)$.

For a ring R,

$$\mathrm{WCart}(R)$$

is defined as the empty category if p is not nilpotent in R, and, if R is p-nilpotent, is the groupoid whose objects are Cartier-Witt divisors on R,

$$(I \xrightarrow{\alpha} W(R))$$

i.e., generalized Cartier divisors on $W(R)$ such that (Zariski locally over $\mathrm{Spec}(R)$) α maps I to $\mathrm{WCart}_0(R)$ ($\Leftrightarrow \mathrm{Im}(I \to W(R) \to R)$ is nilpotent and $\mathrm{Im}(I \to W(R) \xrightarrow{\delta} W(R))$ generates the unit ideal).

The attracting property of WCart

Let (A, I) be a bounded[4] prism. Then the formal scheme $\mathrm{Spf}(A)$ (with the (p, I)-adic topology) canonically maps to the formal stack WCart by a map

$$\rho_{(A,I)} : \mathrm{Spf}(A) \to \mathrm{WCart}$$

defined as follows. For a point $f : A \to R$ of $\mathrm{Spf}(A)$ with value in a (p-nilpotent) ring R, f uniquely lifts to a δ-map $\widetilde{f} : A \to W(R)$, by which the inclusion $I \subset A$ induces a generalized Cartier divisor

$$\rho_{(A,I)}(f) = (I \otimes_A W(R) \xrightarrow{\alpha} W(R)) \in \mathrm{WCart}(R).$$

Then: $f \mapsto \rho_{(A,I)}(f)$ defines $\rho_{(A,I)}$.

[4] This ensures that A is classically (p, I)-complete.

Stacky description of prismatic crystals

The (∞-)category of quasi-coherent complexes on WCart,

$$D(\mathrm{WCart}) := \varprojlim_{\mathrm{Spec}(R) \to \mathrm{WCart}} D(R)$$

(sometimes denoted $D_{\mathrm{qc}}(\mathrm{WCart})$) is by definition the inverse limit of the categories $D(R)$ indexed by the category of points of WCart, i.e. objects $\mathcal{F}$ are coherent rules

$$((I \to W(R)) \in \mathrm{WCart}(R)) \mapsto \mathcal{F}((I \to W(R)) \in D(R).$$

For a bounded prism (A, I), $\rho_{(A,I)} : \mathrm{Spf}(A) \to \mathrm{WCart}$, induces a pull-back map

$$\rho^*_{(A,I)} : D(\mathrm{WCart}) \to D(\mathrm{Spf}(A)) = \widehat{D}(A).$$

For variable (A, I) these maps define a functor

$$D(\mathrm{WCart}) \to \widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(\mathbf{Z}_p)_{\Delta}, \mathcal{O})$$

where the right-hand side is the category of prismatic crystals on the absolute prismatic site $\mathrm{Spf}(\mathbf{Z}_p)_{\Delta}$ (BL, 3.3.5).

Theorem 2 (BL, Prop. 3.3.5). The functor

$$D(\mathrm{WCart}) \to \widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(\mathbf{Z}_p)_{\mathbb{\Delta}}, \mathcal{O})$$

is an equivalence.

In a sense, the Cartier-Witt attractor plays the role of a final object for the site $\mathrm{Spf}(\mathbf{Z}_p)_{\mathbb{\Delta}}$.

Proof. Use the prism $(A^0, I_0 := (x_0))$, where A^0 is the coordinate ring of WCart_0, and the Zariski cover $\mathrm{Spf}(A^0) \to \mathrm{WCart}$.

More generally, for any (bounded) p-adic formal scheme X, there is defined a formal stack

$$\mathrm{WCart}_X$$

over $\mathbf{Z}_p$, called the Cartier-Witt stack of X, which depends functorially on X. For $X = \mathrm{Spf}(W(k))$ as above,

$$\mathrm{WCart}_{\mathrm{Spf}(W(k))} = \mathrm{Spf}(W(k)) \times_{\mathrm{Spf}(\mathbf{Z}_p)} \mathrm{WCart}$$

and the analogue of Th. 2 holds.

For R a p-nilpotent ring, $\mathrm{WCart}_X(R)$ is the groupoid

$$\mathrm{WCart}_X(R) = \{(I \overset{\alpha}{\to} W(R)), x \in X(W(R)/^L I\}),$$

where $(I \overset{\alpha}{\to} W(R))$ is a Cartier-Witt divisor on R, and x a point of X with value in the animated ring $W(R)/^L I$ defined by $(I \overset{\alpha}{\to} W(R))$.

Examples.

- $\mathrm{WCart}_{\mathrm{Spf}(\mathbf{Z}_p)} = \mathrm{WCart}$
- $\mathrm{WCart}_{\mathrm{Spec}(k)} = \mathrm{Spf}(W(k))$.

The construction $X \mapsto \mathrm{WCart}_X$ leads to a theory of prismatization, developed in [BL1].

4. The Hodge-Tate stack

The category $D(\mathrm{WCart})$ is hard to describe “concretely”, but It turns out that WCart contains an effective Cartier divisor, the Hodge-Tate divisor $\mathrm{WCart}^{\mathrm{HT}}$, whose category of quasi-coherent objects on it has a simple description.

The Hodge-Tate divisor

Definition. The Hodge-Tate divisor is the closed substack

$$\mathrm{WCart}^{\mathrm{HT}} \hookrightarrow \mathrm{WCart},$$

whose R-points consist of Cartier-Witt divisors $I \xrightarrow{\alpha} W(R)$ such that the composition $I \xrightarrow{\alpha} W(R) \to R$ is zero.
In other words, it's the fibre product

$$\begin{tikzcd}
\mathrm{WCart}^{\mathrm{HT}} \arrow[r] \arrow[d] & \mathrm{WCart} = [\mathrm{WCart}_0 / W^*] \arrow[d] \\
B\mathbf{G}_m = [\{0\}/\mathbf{G}_m] \arrow[r] & [\widehat{\mathbf{A}}^1/\mathbf{G}_m]
\end{tikzcd}$$

where the right vertical map is induced by the projection $(a_0, a_1, \cdots) \mapsto a_0$.

Equivalently,

$$\mathrm{WCart}^{\mathrm{HT}} \xrightarrow{\sim} [VW^*/W^*].$$

Thus, $R \mapsto (V(1) \in VW^*(R))$ yields a canonical point, the Hodge-Tate point,

$$\eta := V(1) : \mathrm{Spf}(\mathbf{Z}_p) \to \mathrm{WCart}^{\mathrm{HT}}.$$

Theorem 3 ([Dr], [BL]). The HodgeTate point is a flat cover and induces an isomorphism

$$BW^*[F] := [\mathrm{Spf}(\mathbf{Z}_p)/W^*[F]] \xrightarrow{\sim} \mathrm{WCart}^{\mathrm{HT}},$$

where $W^*[F]$ is the stabilizer of η, i.e., the group scheme (over $\mathbf{Z}_p$)

$$W^*[F] := \mathrm{Ker}(F : W^* \to W^*).$$

Proof. By $xVy = V(Fx.y)$ and faithful flatness of $F : W^* \to W^*$, $VW^* = W^*.V(1)$, hence
$\mathrm{WCart}^{\mathrm{HT}} = \mathrm{Cone}(W^* \xrightarrow{F} W^*) = BW^*[F]$.
The left vertical map in above cartesian square is thus identified to

$$BW^*[F] \to B\mathbf{G}_m$$

Main results ([BL], [Dr])

• Identification of Hodge-Tate crystals with quasi-coherent complexes on $\mathrm{WCart}^{\mathrm{HT}}$

• Identification of $W^*[F]$ with $\mathbf{G}_m^\sharp$, PD-envelope at 1 of $\mathbf{G}_m$, and description as an extension

$$0 \to \mu_p \to \mathbf{G}_m^\sharp \to \mathbf{G}_a^\sharp \to 0,$$

where $\mathbf{G}_a^\sharp$ = PD-envelope of $\mathbf{G}_a$ at 0

• Identification of $D(\mathrm{WCart}^{\mathrm{HT}})$ with the category of Sen complexes, i.e. the full subcategory of the (∞)-category $\widehat{D}(\mathbf{Z}_p[\Theta])$ of objects M of $\widehat{D}(\mathbf{Z}_p)$ endowed with an endomorphism Θ such that Θ (the Sen operator) has the property that $\Theta^p - \Theta$ on $H^*(M \otimes^L \mathbf{F}_p)$ is locally nilpotent.

(Discussed in Section 5.)

• Identification of Hodge-Tate crystals with quasi-coherent complexes on $\mathrm{WCart}^{\mathrm{HT}}$

Similarly to $D(\mathrm{WCart})$ define

$$D(\mathrm{WCart}^{\mathrm{HT}}) := \varprojlim_{\mathrm{Spec}(R) \to \mathrm{WCart}^{\mathrm{HT}}} D(R).$$

If (A, I) is a bounded prism, then $\rho_{(A,I)} = \mathrm{Spf}(A) \to \mathrm{WCart}$ restricts to

$$\rho^{\mathrm{HT}}_{(A,I)} : \mathrm{Spf}(A/I) \to \mathrm{WCart}^{\mathrm{HT}},$$

$$\rho_{(A,I)}(f : A/I \to R) = (I \otimes_A W(R) \xrightarrow{\alpha} W(R)) \in \mathrm{WCart}^{\mathrm{HT}}(R).$$

From

$$\rho^{\mathrm{HT}}_{(A,I)} : \mathrm{Spf}(A/I) \to \mathrm{WCart}^{\mathrm{HT}},$$

get pull-back map

$$(\rho^{\mathrm{HT}}_{(A,I)})^* : D(\mathrm{WCart}^{\mathrm{HT}}) \to \widehat{D}(A/I),$$

and functor

$$D(\mathrm{WCart}^{\mathrm{HT}}) \to \varprojlim_{(A,I)} \widehat{D}(A/I) = \widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(\mathbf{Z}_p)_{\mathbb{\Delta}}, \overline{\mathcal{O}}),$$

where the right hand side is the (∞-)category of p-complete Hodge-Tate crystals (section 2, end). The above classification of p-complete crystals, restricted to Hodge-Tate crystals, yields:

Theorem 4. The above functor is an equivalence:

$$D(\mathrm{WCart}^{\mathrm{HT}}) \xrightarrow{\sim} \widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(\mathbf{Z}_p)_{\mathbb{\Delta}}, \overline{\mathcal{O}})$$

More generally, given a p-adic formal scheme X, one defines the Hodge-Tate divisor $\mathrm{WCart}_X^{\mathrm{HT}}$ by the pull-back square:

$$\begin{tikzcd}
\mathrm{WCart}_X^{\mathrm{HT}} \arrow[r] \arrow[d] & \mathrm{WCart}_X \arrow[d] \\
\mathrm{WCart}^{\mathrm{HT}} \arrow[r] & \mathrm{WCart}
\end{tikzcd}.$$

Examples. For k as above,

$$\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spec}(k)} = \mathrm{Spec}(k),$$

$$\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W(k))} = \mathrm{Spf}(W(k)) \times_{\mathrm{Spf}(\mathbf{Z}_p)} \mathrm{WCart}^{\mathrm{HT}}.$$

and we have the equivalence

$$D(\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf(W(k))}}) \xrightarrow{\sim} \widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(W(k))_{\Delta}, \overline{\mathcal{O}}),$$

Corollary. There are canonical equivalences

$$\widehat{D}_{\mathrm{crys}}(\mathrm{Spf}(W(k))_{\mathbb{\Delta}}, \overline{\mathcal{O}}) \overset{\sim}{\leftarrow} D(\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W(k))}) \overset{\eta^*}{\to} D((BW^*[F])_{\mathrm{Spf}(W(k))}),$$

$$\mathcal{E} \mapsto \mathcal{E}_\eta := \eta^*(\mathcal{E})$$

where

• $\eta = V(1) : \mathrm{Spf}(W(k)) \to \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W(k))}$ is the canonical point defined above

• a Hodge-Tate crystal is identified with the corresponding quasi-coherent complex on the Hodge-Tate stack.

• an object $\mathcal{E}$ of $D(\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W(k))}) = D((BW^*[F])_{\mathrm{Spf}(W(k))})$ is identified by $\mathcal{E} \mapsto \mathcal{E}_\eta := \eta^*(\mathcal{E})$ with a pair of an object $E \in \widehat{D}(\mathrm{Spf}(W(k))$ and an action α of $\mathbf{G}_m^\sharp$ on it.

The group scheme $\mathbf{G}_m^\sharp$

Proposition ([Dr, 3.2.6], [BL, 3.4.11, 3.5.18]) (i) The composite $W[F] \to W \to \mathbf{G_a}$ induces an isomorphism

$$W[F] \xrightarrow{\sim} \mathbf{G}_a^\sharp = \mathrm{Spec}(D_{(t)}\mathbf{Z}_p[t]) = \mathrm{Spec}(\Gamma_{\mathbf{Z}_p}(\mathbf{Z}_p t)).$$

(ii) The composite $W^*[F] \to W^* \to \mathbf{G}_m$ induces an isomorphism

$$W^*[F] \xrightarrow{\sim} \mathbf{G}_m^\sharp = \mathrm{Spec}(D_{(t-1)}(\mathbf{Z}_p[t, t^{-1}]).$$

(iii) There is an exact sequence of group schemes (over $\mathbf{Z}_p$)

$$0 \to \mu_p \xrightarrow{[.]} \mathbf{G}_m^\sharp \xrightarrow{\log(-)} \mathbf{G}_a^\sharp \to 0,$$

split over $\mathbf{F}_p$, where

$$\mathbf{G}_a^\sharp := \mathrm{Spec}(\mathbf{Z}_p\langle t\rangle)(\xrightarrow{\sim} W[F])$$

is the PD-envelope of $\mathbf{G}_a$ at the origin.

Proof. Main point is (i). Drinfeld's argument: use description of $\mathbf{Z}_p\langle t\rangle$ by generators $u_n = t^{p^n}/p^{\frac{p^n-1}{p-1}}$ and relations $u_n^p = pu_{n+1}$, and Joyal's theorem to the effect that the coordinate ring $B = \Gamma(W, \mathcal{O})$ of W is the free δ-ring on one indeterminate y_0, i.e., is the polynomial ring

$$B = \mathbf{Z}_p[y_0, y_1, \cdots],$$

with $y_n = \delta^n(y_0)$.

5. Sen operators, Hodge diffraction

Let $\mathcal{E} \in D(\mathrm{WCart}^{\mathrm{HT}}) = D(B\mathbf{G}_m^\sharp)$[5], that we identify with the pair of an object $E = \mathcal{E}_\eta \in \widehat{D}(\mathrm{Spf}(W(k))$ and an action $\alpha : \mathbf{G}_m^\sharp \to \mathrm{Aut}(E)$. Consider the induced infinitesimal action

$$\mathrm{Lie}(\alpha) : \mathrm{Lie}(\mathbf{G}_m^\sharp) \to \mathrm{End}(E),$$

where $\mathrm{Lie}(\mathbf{G}_m^\sharp) = \mathbf{G}_m^\sharp(\mathrm{Spf}(W(k))[\varepsilon]/(\varepsilon^2))$.
In particular, the point $1 + [\varepsilon] \in \mathrm{Lie}(\mathbf{G}_m^\sharp)$ gives an endomorphism

$$\Theta_{\mathcal{E}} \in \mathrm{End}(E)$$

called the Sen operator.

The Sen operators satisfy a Leibniz rule

$$\Theta_{\mathcal{E}\otimes\mathcal{F}} = \Theta_{\mathcal{E}} \otimes \mathrm{Id}_{\mathcal{F}} + \mathrm{Id}_{\mathcal{E}} \otimes \Theta_{\mathcal{F}}.$$

---

[5] In this section we work over $W(k)$ and in general omit the subscript $\mathrm{Spf}(W(k))$.

Examples. (1) $\Theta_{\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}} = 0$.

(2) For the (Hodge-Tate) Breuil-Kisin twist $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{1\}$, i.e., the line bundle on $\mathrm{WCart}^{\mathrm{HT}}$ defined by $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I} : (A, I) \mapsto I$:

$$\Theta_{\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{1\}} = \mathrm{Id}.$$

Hence $\Theta_{\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{n\}} = n\mathrm{Id}$.

(Note: The Hodge-Tate crystal $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{1\}$ is induced on the Hodge-Tate divisor from the (crystalline) Breuil-Kisin line bundle $\mathcal{O}_{\mathrm{WCart}}\{1\}$), a prismatic F-crystal [BL, 3.3.8] satisfying $\varphi^*\mathcal{O}_{\mathrm{WCart}}\{1\} \xrightarrow{\sim} \mathcal{I}^{-1} \otimes \mathcal{O}_{\mathrm{WCart}}\{1\}$).

(3) The cartesian square

$$\begin{tikzcd} \mathbf{G}_m^\sharp \arrow[r] \arrow[d] & \mathrm{Spf}(W(k)) \arrow[d, "\eta"'] \\ \mathrm{Spf}(W(k)) \arrow[r, "\eta"] & \mathrm{WCart}^{\mathrm{HT}} \end{tikzcd}$$

yields

$$\eta^*\eta_*\mathcal{O} = \widehat{\mathcal{O}}^\sharp_{\mathbf{G}_m},$$

where the right hand side denotes the p-completion of the coordinate ring $D_{(t-1)}(W(k)[t, t^{-1}])$ of $\mathbf{G}_m^\sharp$. One has:

$$\Theta_{\widehat{\mathcal{O}}^\sharp_{\mathbf{G}_m}} = t\partial/\partial t.$$

Denote by

$$\widehat{D}(W(k)[\Theta])$$

the category of pairs (M, Θ_M) where M is a p-complete object of $D(W(k))$ and Θ_M an endomorphism of M.

Theorem 5 [BL, 3.5.8]. The functor

$$D(\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W(k))}) \to \widehat{D}(W(k)[\Theta]), \mathcal{E} \mapsto (\mathcal{E}_\eta, \Theta_{\mathcal{E}})$$

is fully faithful and its essential image consists of pairs (M, Θ_M) such that $(\Theta_M)^p - \Theta_M$ is locally nilpotent[6] on $H^*(M \otimes^L_{W(k)} k)$ (such pairs are called Sen complexes).

[6] i.e., for each $x \in H^i$, there exists $n(x)$ such that $(\Theta_M^p - \Theta_M)^n.x = 0$ for $n \geqslant n(x)$.

Proof. Main points: (i) A fixed point formula: for $\mathcal{E} \in D(\mathrm{WCart}^{\mathrm{HT}})$,

$$\mathcal{E} \xrightarrow{\sim} (\eta_* \eta^* \mathcal{E})^{\Theta=0}$$

(ii) Dévissage (using the co-regular representation of $\widehat{\mathcal{O}}^{\sharp}_{\mathbf{G}_m}$) showing that $D(\mathrm{WCart}^{\mathrm{HT}})$ is generated, under shifts and colimits, by the Breuil-Kisin twists $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{n\}$ ($n \geqslant 0$).

The diffracted Hodge complex $\Omega^{\not{D}}_{X/W(k)}$

Let's come back to our formal smooth $f : X \to \mathrm{Spf}(W(k))$.

(a) Assume first that X is affine, $X = \mathrm{Spf}(R)$.

Denote by

$$\Omega^{\not{D}}_{R/W(k)} \in \widehat{D}(\mathrm{Spf}(W(k))[\Theta])$$

the Sen complex associated with the p-complete Hodge-Tate crystal over $\mathrm{Spf}(W(k))$

$$(A, I) \mapsto (Rf_* \overline{\mathcal{O}}_{\mathbb{A}})_{(A,I)} = R\Gamma_{\overline{\mathbb{A}}}(X_{\overline{A}}/A) := R\Gamma((X_{\overline{A}}/A)_{\mathbb{A}}, \overline{\mathcal{O}}) \in \widehat{D}(\overline{A}).$$

This Sen complex is called the diffracted Hodge complex of R/k.

The canonical truncation filtration of $R\Gamma_{\overline{\mathbb{A}}}(X_{\overline{A}}/A)$ for $(A, I) \in \mathrm{Spf}(W(k)_{\mathbb{A}}$ defines a canonical increasing, multiplicative filtration of $\Omega^{\not{D}}_{R/W(k)}$, called the conjugate filtration, which is stable under Θ,

$$\mathrm{Fil}^{\mathrm{conj}}_{\bullet} = (\mathrm{Fil}^{\mathrm{conj}}_0 \to \mathrm{Fil}^{\mathrm{conj}}_1 \to \cdots).$$

It follows from the Hodge-Tate comparison theorem and the smoothness of $R/W(k)$ that

$$\mathrm{gr}_i^{\mathrm{conj}} = \Omega^i_{R/W(k)}[-i]\{-i\}$$

In particular, $\mathrm{Fil}_0^{\mathrm{conj}} = R$, so that $\Omega^{\not{D}}_{R/W(k)}$ can be promoted to a filtered object of $\widehat{D}(R)[\Theta]$, which is perfect as a filtered object of $\widehat{D}(R)$.

By Examples (1) and (2) above, we have

$$\Theta | H^i(\Omega^{\not{D}}_{R/W(k)}) = -i.$$

(b) For a general formal smooth $f : X \to \mathrm{Spf}(W(k))$, the $\Omega^{\not{D}}_{R/W(k)}$ patch to a filtered perfect complex in $D(X, \mathcal{O}_X)$, called the diffracted Hodge complex of $X/W(k)$

$$\Omega^{\not{D}}_{X/W(k)},$$

equipped with a Sen operator Θ satifying

$$\Theta | H^i(\Omega^{\not{D}}_{X/W(k)}) = -i.$$

(which implies (the already known) fact that

$\Theta^p - \Theta$ on $H^*(\Omega^{\not{D}}_{X/W(k)} \otimes^L k)$ is nilpotent, and even zero (as $H^*(\Omega^{\not{D}}_{X/W(k)})$ is locally free of finite type over X).

Remarks. (1) The Hodge complex $\Omega^*_{X/W(k)} := \oplus_i \Omega^i_{X/W(k)}[-i]$ and the diffracted one $\Omega^{\not{D}}_{X/W(k)}$ are both filtered perfect complexes in $D(X, \mathcal{O}_X)$: the former one, with the trivial filtration, with $\mathrm{gr}^i = \Omega^i_{X/W(k)}[-i]$, the latter one with the canonical filtration, with $\mathrm{gr}_i = \Omega^i_{X/W(k)}[-i]\{-i\}$ (and the additional structure Θ). Bhatt and Lurie view this deviation and enrichment as a diffraction phenomenon, like a wave being diffracted by a slit ($\eta : \mathrm{Spf}(W(k)) \to \mathrm{WCart}^{\mathrm{HT}}$).

(2) Let $K := W(k)[1/p]$ and $C := \widehat{\overline{K}}$. It is shown in [BL, 3.9.5, 4.7.22] that by extending the scalars to $\mathcal{O}_C$, and using the prismatic - étale comparison theorem, Θ corresponds to the classical Sen operator on the (semilinear) representation $C \otimes_{W(k)} H^*(X_{\overline{K}}, \mathbf{Z}_p)$ of $\mathrm{Gal}(\overline{K}/K)$ and (for $X/W(k)$ proper and smooth) yields the Hodge-Tate decomposition

$$C \otimes H^n(X_{\overline{K}}, \mathbf{Z}_p) \xrightarrow{\sim} \oplus_i C(-i) \otimes_{W(k)} H^{n-i}(X, \Omega^i_{X/W(k)}).$$

End of proof of Th. 1.

Recall: $Y := X \otimes_{W(k)} k$. Define

$$\Omega^{\not{D}}_{Y/k} := \Omega^{\not{D}}_{X/W(k)} \otimes^{L}_{W(k)} k \in D(Y, \mathcal{O}_Y),$$

and let again Θ denote the endomorphism induced by the Sen operator of $\Omega^{\not{D}}_{X/W(k)}$.

As we already know that

(i) $H^i(\Omega^{\not{D}}_{Y/k}) \xrightarrow{\sim} \Omega^i_{Y/k}$ canonically,

(ii) Θ is a derivation, and acts by $-i$ on H^i,

it remains to construct the isomorphism (in $D(Y', \mathcal{O}_{Y'})$)

$$\varepsilon : \varphi^* \Omega^{\not{D}}_{Y/k} \xrightarrow{\sim} F_* \Omega^{\bullet}_{Y/k}, \tag{0.2}$$

with the property:

(iii) ε is multiplicative and (via (i)) induces the Cartier isomorphism $C^{-1} : \Omega^i_{Y'/k} \xrightarrow{\sim} H^i(F_* \Omega^{\bullet}_{Y/k})$ on H^i.

Interlude: Sen complexes and evaluation of Hodge-Tate crystals

A preliminary is needed for the construction of ε.

Recall that if $\mathcal{E}$ is a (p-complete) Hodge-Tate crystal on $\mathrm{Spf}(W(k))_{\mathbb{\Delta}}$, the corresponding Sen complex (E, Θ) is defined by

$$E = \eta^* \mathcal{E},$$

where $\mathcal{E}$ is identified with an object of $D(\mathrm{WCart}^{\mathrm{HT}}) = D(B\mathbf{G}_m^\sharp)$, and

$$\eta : \mathrm{Spf}(W(k)) \to B\mathbf{G}_m^\sharp$$

is the point $V(1)$, corresponding to the trivial $\mathbf{G}_m^\sharp$-torsor on $\mathrm{Spf}(W(k))$.

Let $(A, I) \in \mathrm{Spf}(W(k))_{\Delta}$. Consider the canonical map

$$\rho^{\mathrm{HT}}_{(A,I)} : \mathrm{Spf}(\overline{A}) \to \mathrm{WCart}^{\mathrm{HT}} = B\mathbf{G}_m^\sharp$$

It corresponds to a $\mathbf{G}_m^\sharp$-torsor $\mathcal{P} = \mathcal{P}_{(A,I)}$ over $\mathrm{Spf}(\overline{A})$, and this torsor is trivial if and only if one can fill in the diagram (of A-linear maps)

$$\begin{tikzcd} I \arrow[d] & W(\overline{A}) \arrow[d, "V(1)"'] \\ A \arrow[r, "\psi_{(A,I)}"'] & W(\overline{A}) \end{tikzcd}$$

with a top horizontal A-linear map $\xi : I \to W(\overline{A})$ making the square commute, where $\psi_{(A,I)}$ is the unique lift of $A \to \overline{A}$ compatible with δ.

We'll say that (A, I) is neutral if $\rho^{\mathrm{HT}}_{(A,I)}$ factors through η, i.e., $\mathcal{P}_{(A,I)}$ is trivial). If (A, I) is neutral, then

$$\mathcal{E}(\overline{A}) \xrightarrow{\sim} \overline{A} \otimes_{W(k)} \mathcal{E}_\eta$$

in $\widehat{D}(\overline{A})$.

Consider the q-de Rham prism $Q := (\mathbf{Z}_p[[q-1]], ([p]_q), \varphi(q) = q^p)$ on which $i \in \mathbf{F}_p^*$ acts by $q \mapsto q^{[i]}$ ($[i] \in \mathbf{Z}_p^*$ the Teichmüller representative). Let

$$Q_0 := Q^{\mathbf{F}_p^*}$$

By [BL, 3.8.6]

$$Q_0 = (\mathbf{Z}_p[[\widetilde{p}]], (\widetilde{p}), \varphi(q) = q^p),\ \widetilde{p} := \sum_{i \in \mathbf{F}_p} q^{[i]}).$$

and the prism $(A, I) = W(k) \otimes_{\mathbf{Z}_p} Q_0$ is neutral.

Remark (Gabber). The element $p - [p] \in W(\mathbf{Z}_p)$ is of the form Vx, for x with ghost coordinates

$$w(x) = (1 - p^{p-1}, 1 - p^{p^2-1}, \cdots),$$

and x is in the image of F if and only if p is odd. Therefore the Breuil-Kisin prism $(A, I) = (W(k)[[u]], (p - u), u \mapsto u^p)$ has $A/I = W(k)$, but is neutral if and only if p is odd.

Construction of ε.

Applying the above to the Hodge-Tate crystal $\mathcal{E} = Rf_*\overline{\mathcal{O}}_{\mathbb{\Delta}}$ for $f : \mathrm{Spf}(R) \to \mathrm{Spf}(W(k))$, and the prism $(A, I) = W(k) \otimes_{\mathbf{Z}_p} Q_0$, with $\overline{A} = W(k)$ we find

$$\Omega^{\not{D}}_{R/W(k)} \xrightarrow{\sim} (Rf_*\overline{\mathcal{O}}_{\mathbb{\Delta}})_{(A,I)}$$

in $\widehat{D}(R)$, and then, for a general formal smooth $X/W(k)$,

$$\Omega^{\not{D}}_{X/W(k)} \xrightarrow{\sim} \overline{\mathbb{\Delta}}_{X/A}$$

in $D(X, \mathcal{O}_X)$. The de Rham comparison theorem (dR) thus provides a multiplicative isomorphism (in $D(X, W(k))$)

$$\varphi_A^*\Omega^{\not{D}}_{X/W(k)} \xrightarrow{\sim} \varphi_*\Omega^{\bullet}_{X/W(k)},$$

which, by reduction mod p yields the desired isomorphism (in $D(Y', \mathcal{O}_{Y'})$)

$$\varepsilon : \varphi^*\Omega^{\not{D}}_{Y/k} \xrightarrow{\sim} F_*\Omega^{\bullet}_{Y/k}. \qquad (0.2)$$

Remarks on the mod p^2 lifted case

A formal smooth lifting X of Y over $\mathrm{Spf}(W_2(k))$ instead of over $\mathrm{Spf}(W(k))$ gives rise to a similar story and yields the general case of Th. 1. Note, however, that

$$\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W_n(k))} \not\xrightarrow{\sim} (B\mathbf{G}_m^\sharp)_{\mathrm{Spf}(W_n(k))}.$$

(e. g., $\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spec}(k)} = \mathrm{Spec}(k)$). For all $n \geqslant 1$,

$$\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W_n(k))} = [\mathrm{Spf}(W_n(k))^{\not{D}}/\mathbf{G}_m^\sharp],$$

where $\mathrm{Spf}(W_n(k))^{\not{D}}$ is the diffracted Hodge stack of $\mathrm{Spf}(W_n(k))$, defined by the fiber square

$$\begin{tikzcd}
\mathrm{Spf}(W_n(k))^{\not{D}} \arrow[r] \arrow[d] & \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W_n(k))} \arrow[d] \\
\mathrm{Spf}(W(k)) \arrow[r, "\eta"] & \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spf}(W(k))}.
\end{tikzcd}$$

In particular [BL1, 5.15], for $n \geqslant 2$,

$$\mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spec}(W_n(k))} \times_{\mathrm{Spf}(W_n(k))} \mathrm{Spec}(k) \xrightarrow{\sim} [\mathbf{G}_a^\sharp / \mathbf{G}_m^\sharp]_{\mathrm{Spec}(k)}.$$

Therefore the composite

$$\mathrm{Spec}(k) \to \mathrm{Spf}(W(k)) \xrightarrow{\eta} (B\mathbf{G}_m^\sharp)_{\mathrm{Spf}(W(k)}$$

factors through a unique map

$$\eta_2 : \mathrm{Spec}(k) \to \mathrm{WCart}^{\mathrm{HT}}_{\mathrm{Spec}(W_2(k))} \times_{\mathrm{Spec}(W_2(k))} \mathrm{Spec}(k) = [\mathbf{G}_a^\sharp / \mathbf{G}_m^\sharp]_{\mathrm{Spec}(k)}$$

a section of $[\mathbf{G}_a^\sharp / \mathbf{G}_m^\sharp]_{\mathrm{Spec}(k)}$, whose automorphism group is $(\mathbf{G}_m^\sharp)_{\mathrm{Spec}(k)}$.

This suffices to carry over the arguments to the mod p^2 case.

6. An alternate approach: endomorphisms of the de Rham functor (after Li-Mondal, Mondal)

Let Y/k be smooth. The construction of a Sen structure on $F_*\Omega^\bullet_{Y/k}$ provided by a formal smooth $X/W_2(k)$ lifting Y uses the *deus ex machina* WCart. One can ask:

(1) Can one understand this hidden structure more concretely?

(2) Can one bypass WCart to construct it?

While (1) remains largely open, Li-Mondal [LM] have recently given an independent proof of Th. 1, which doesn't use prismatization, but instead, a certain ring stack $\mathbf{G}_a^{\mathrm{dR}}$ over $W(k)$, the de Rham stack (an avatar of WCart), which generates the de Rham cohomology functor.

It was subsequently shown by Mondal [M] that this stack is not a *deus ex machina*, but, in fact, can be reconstructed from the de Rham cohomology functor.

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de Rham cohomology functor

(Drinfeld, Li-Mondal, Bhatt) $\Uparrow \Downarrow$ (Mondal)

The de Rham stack $\mathbf{G}_a^{\mathrm{dR}}$

$\Downarrow$ (Li-Mondal)

Endomorphisms of the de Rham functor

$\Downarrow$

Theorem 1

The de Rham stack

The de Rham stack is the ring stack over $\mathrm{Spf}(W(k))$

$$\mathbf{G}_a^{\mathrm{dR}} := [\mathbf{G}_a/\mathbf{G}_a^\sharp]$$

where $\mathbf{G}_a^\sharp = W[F] = \mathrm{Spec}(W(k)\langle t\rangle)$ is viewed as a quasi-ideal in $\mathbf{G}_a$ via the canonical map

$$\mathbf{G}_a^\sharp \to \mathbf{G}_a$$

induced by the projection $W \to \mathbf{G}_a$, $x \mapsto x_0$, corresponding to $W[t] \to W\langle t\rangle$. [7] Points of $\mathbf{G}_a^{\mathrm{dR}}$ with value in a p-complete $W(k)$-algebra R are the groupoid underlying the animated $W(k)$-algebra

$$\mathbf{G}_a^{\mathrm{dR}}(R) = (\mathbf{G}_a^\sharp(R) \to \mathbf{G}_a(R)).$$

---

[7] (an analogue of the Simpson stack $[\mathbf{G}_a/\widehat{\mathbf{G}_a}]$ in characteristic zero)

Relations with WCart and de Rham cohomology

- Reconstruction of de Rham cohomology

(Bhatt) For $X/\mathrm{Spf}(W(k))$ formal smooth, define the de Rham stack of X

$$X^{\mathrm{dR}}$$

by $X^{\mathrm{dR}}(R) = X(\mathbf{G}_a^{\mathrm{dR}}(R))$ on p-complete $W(k)$-algebras R, i.e., for $X = \mathrm{Spf}(A)$, $X^{\mathrm{dR}}(R) = \mathrm{Hom}(A, \mathbf{G}_a^{\mathrm{dR}}(R))$, Hom taken in the category of animated $W(k)$-algebras.

Theorem 6 (Bhatt, Li-Mondal)[8] There is a functorial isomorphism

$$R\Gamma_{dR}(X/W(k)) = R\Gamma(X^{dR}, \mathcal{O})$$

The definition of X^{dR} is a special case of Li-Mondal's theory of unwinding [LM].

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[8](elaborating on a theorem of Drinfeld [Dr0,Th. 2.4.2])

- Relation with WCart and the de Rham point

(a) (Drinfeld) $\mathbf{G}_a^{\mathrm{dR}} \xrightarrow{\sim} [W \xrightarrow{p} W]$

(b) Consider the de Rham point

$$\rho_{\mathrm{dR}} = \rho_{(\mathbf{Z}_p,(p))} : \mathrm{Spf}(\mathbf{Z}_p) \to \mathrm{WCart}$$

(corresponding to $p = (p, 1 - p^{-1}, \cdots) \in \mathrm{WCart}_0(\mathbf{Z}_p)$).

By Drinfeld's formula above ρ_{dR} "generates" the de Rham stack, and, thanks to the prismatic de Rham comparison theorem yields, by pull-back, another proof of Th. 6 [BL, Prop. 5.4.8].

Endomorphisms of the de Rham functor

By unwinding and using that $\mathbf{G}_a^{\mathrm{dR}}$ is an affine stack in the sense of Toën [T] Li-Mondal [LM] show that $\mathbf{G}_a^{\mathrm{dR}}$ controls the endomorphisms of the de Rham functor. In particular, they prove:

Theorem 7 [LM, Th. 4.23] For a k-algebra B, let $\mathrm{CAlg}(D(B))$ denote the category of commutative algebra objects in the (∞-) category $D(B)$. Consider the group functor on the category of k-algebras defined by

$$F : B \mapsto \mathrm{Aut}(\widetilde{R} \mapsto \Omega^{\bullet}_{\widetilde{R} \otimes_{W_2(k)} k/k} \otimes_k B \in \mathrm{CAlg}(D(B))),$$

where $\widetilde{R}$ runs through the smooth $W_2(k)$-algebras. Then F is represented by $\mathbf{G}^{\sharp}_{m,k}$.

Applying Th. 7 for the Hopf algebra $B = \Gamma(\mathbf{G}^{\sharp}_{m,k}, \mathcal{O})$, Li-Mondal deduce the (functorial in $\widetilde{R}$) action of $\mathbf{G}^{\sharp}_{m,k}$ on $\Omega^{\bullet}_{\widetilde{R}\otimes k/k}$, and, finally, the Sen structure given in Th. 1.

As a bonus, they prove:

Corollary (1) There is a unique splitting

$$\mathcal{O}_{\widetilde{X}'_k} \oplus \Omega^1_{\widetilde{X}'_k}[-1] \xrightarrow{\sim} \tau^{\leqslant 1} F_* \Omega^{\bullet}_{\widetilde{X}_k/k},$$

inducing C^{-1} on H^i, and functorial in the smooth scheme $\widetilde{X}/W_2(k)$. In particular, the splittings constructed by Drinfeld, Bhatt-Lurie and Li-Mondal coincide.

(2) There is no functorial splitting $F_*\Omega^{\bullet}_{\widetilde{X}_k/k} \xrightarrow{\sim} \oplus_i H^i(F_*\Omega^{\bullet}_{\widetilde{X}_k/k})[-i]$ as functors to $\mathrm{CAlg}(D(k))$ from smooth schemes $\widetilde{X}$ over $W_2(k)$.

Remark. Part (2) was proved independently by Mathew.

Reconstruction of the de Rham stack from de Rham cohomology

The functor $R \mapsto \Omega^\bullet_{R/k}$ from the category of smooth k-algebras to $\mathrm{CAlg}(D(k))$ extends by left Kan extension to a functor

$$\mathrm{dR} : \mathrm{ARings}_k \to \mathrm{CAlg}(D(k)),\ R \mapsto L\Omega^\bullet_{R/k},$$

where ARings_k is the category of animated k-algebras.
As dR commutes with colimits, dR has a right adjoint

$$\mathrm{dR}^\vee : \mathrm{CAlg}(D(k)) \to \mathrm{ARing}_k.$$

Let $\mathrm{Alg}_k \subset \mathrm{ARings}_k$ be the full subcategory of usual commutative k-algebras, and

$$\mathrm{dR}_0^\vee : \mathrm{Alg}_k \to \mathrm{ARing}_k$$

be the restriction of $\mathrm{dR}^\vee$ along $\mathrm{Alg}_k \subset \mathrm{ARings}_k \to \mathrm{CAlg}(D(k))$.

Theorem 8. (Mondal). There is a canonical isomorphism

$$\mathrm{dR}_0^\vee \xrightarrow{\sim} (\mathbf{G}_a^{\mathrm{dR}})_k.$$

7. Questions

This theory of diffraction and Sen complexes forms a new territory, which has not yet been much explored. Here are a few questions.

Question 1. Is there a smooth Y/k, liftable to $W_2(k)$, such that

$$(*) \qquad F_*\Omega^\bullet_{Y/k} \not\xrightarrow{\sim} \oplus_i H^i(F_*\Omega^\bullet_{Y/k})[-i]$$

in $D(Y', \mathcal{O}_{Y'})$?

Question already raised in [DI]. A counterexample should have dimension $\geqslant p+1$. By Cor. (2) to Th. 7, there is no splitting (*) of $F_*\Omega^\bullet_{Y/k}$, for $Y = \widetilde{Y} \otimes k$, which is *multiplicative* (i.e., with values in $\mathrm{CAlg}(D(k))$) and *functorial* in the lifting $\widetilde{Y}/W_2(k)$.

Questions 2. Let Y/k be smooth, having a lifting $\widetilde{Y}$ to $W_2(k)$, so that by Th. 1 we have a Sen structure $(\Omega^{\not{D}}_{Y/k}, \Theta, \varepsilon)$ on $F_*\Omega^\bullet_{Y/k}$.

(a) Does there exist a pair $(Y, \widetilde{Y})$ such that, for each $i \in \mathbf{Z}/p\mathbf{Z}$, $\Theta_i \in \mathrm{End}((F_*\Omega^\bullet_{Y/k})_i)$ is non-zero?

(Petrov [P] constructed an example with $\Theta_0|\tau^{\leqslant p}(F_*\Omega^\bullet_{Y/k})_0$ not 0.)

(b) ([BL, 4.7.20]) Is there a bound, independent of $\dim(Y)$ for the orders of nilpotency of the Θ_i's?

(c) The isomorphism classes of lifts $\widetilde{Y}$ form an affine space A under $\mathrm{Ext}^1(\Omega^1_{Y'/k}, \mathcal{O}_{Y'})$. For each $x \in A$, $\Theta_0(x)$, restricted to $\tau^{\leqslant p}(F_*\Omega^\bullet_{Y/k})_0$ is an element $c(x) \in \mathrm{Ext}^p(\Omega^p_{Y'/k}, \mathcal{O}_{Y'})$. Can one explicitly describe the map

$$c : A \to \mathrm{Ext}^p(\Omega^p_{Y'/k}, \mathcal{O}_{Y'})?$$

Question 3. Generalize Sen structures to families, i.e., replace $W_2(k)$ by a parameter space T over $W_2(k)$.

Question 4. (Bhatt) Is there an analogue of the Sen story over other prisms than $(W(k), (p))$? Suppose (A, I) is an absolute (bounded) prism, and $X \to \mathrm{Spf}(A/I)$ formal smooth is lifted to $\widetilde{X}$ formal smooth over $\mathrm{Spf}(A)$ (or just $\mathrm{Spf}(A/I^2)$), does the datum of $\widetilde{X}$ gives extra structure on $\overline{\mathbb{\Delta}}_{X/A} \in D(X, \mathcal{O}_X)$?

Finally, let me mention 3 problems on which there is ongoing work:

(a) Behavior of Θ with respect to the (decreasing) Hodge filtration[9] of $\Omega^{\not{D}}_{Y/k}$ and analogy of $\Theta^p - \Theta$ with a p-curvature. Link with Drinfeld's Σ' [Dr, section 5] and the extended Hodge-Tate stack

$$[\mathbf{G}_a^{\mathrm{dR}}/\mathbf{G}_m]$$

of which $B\mathbf{G}_m^{\sharp}$ is an open substack. Ongoing work by Bhatt-Lurie [BL 4.7.23].

(b) Problem of reconstructing of WCart from prismatic cohomology: generalization of Th. 8 (reconstruction of $\mathbf{G}_a^{\mathrm{dR}}$ from de Rham cohomology). Ongoing work by Mondal.

(c) Derived and log variants. Ongoing work by (Mathew-Yao, Mondal).

[9](deduced from the naive filtration of $F_*\Omega^{\bullet}_{Y/k}$ by the isomorphism $\Omega^{\not{D}}_{Y'/k} \xrightarrow{\sim} F_*\Omega^{\bullet}_{Y/k}$)

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