Periods, Motives and Differential Equations: between Arithmetic and Geometry

on the occasion of Yves André’s 60th birthday

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New perspectives on de Rham cohomology, after Bhatt-Lurie, Drinfeld, et al.

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0. Introduction

$k$ perfect field of char. $p > 0$ ; $X/k$ smooth

\[ X \xleftarrow{\varphi:=F_k} X' \xleftarrow{F} X \]

\[ \Spec(k) \xleftarrow{\varphi:=F_k} \Spec(k) \]

Cartier isomorphism

\[ C^{-1} : \Omega^i_{X'/k} \isom H^i(F_\ast \Omega^\bullet_{X/k}) \]

By [Di] a smooth lifting $\tilde{X}/W_2(k)$ of $X$ gives a decomposition in $D(X', \mathcal{O}_{X'})$

\[ \bigoplus_{0 \leq i < p} \Omega^i_{X'/k}[-i] \isom \tau^{<p} F_\ast \Omega^\bullet_{X/k} \]

inducing $C^{-1}$ on $H^i$. 
Recall the main steps of the proof:

1. $(\tau \geq -1 L\Omega^1_{X'/W(k)})[-1] \xrightarrow{\sim} \tau \leq 1 F_* \Omega^\bullet_{X/k}$ in $D(X', \mathcal{O}_{Y'})$
   (use local liftings of Frobenius)

2. $\tilde{X}/W_2(k)$ lifting $X$ gives splitting of $L\Omega^1_{X/W_2(k)}$
   (elementary cotangent complex theory)

3. from $\tau \leq 1$ to $\tau < p$: multiplicativity
Goal today: give an idea of the proof of the following theorem of Bhatt-Lurie (refining an unpublished, independent one of Drinfeld):

**Theorem 1** [BL1, 5.16]. Let $X/k$ be smooth. A smooth lifting $\tilde{X}$ of $X$ to $W_2(k)$ determines a commutative algebra object\(^1\)

$$\Omega_{\tilde{X}/k}^\phi \in D(X, \mathcal{O}_X),$$

depending functorially on $\tilde{X}$, which is a perfect complex, equipped with an endomorphism $\Theta$ and an isomorphism in $D(X', \mathcal{O}_{X'})$

\[(0.2)\] $$\varepsilon : \varphi^*\Omega_{X/k}^\phi \simto F_*\Omega_{X/k}^\bullet,$$

with the following properties:

(i) $H^i(\Omega_{\tilde{X}/k}^\phi) \simto \Omega_{X/k}^i$ canonical, multiplicative;

(ii) $\Theta$ is a derivation, and acts by $-i$ on $H^i$;

(iii) $\varepsilon$ is multiplicative and induces the Cartier isomorphism

\[C^{-1} : \Omega_{X'/k}^i \simto H^i(F_*\Omega_{X/k}^\bullet) \text{ on } H^i.\]

\(^1\)From now on, derived categories are taken in the derived $\infty$-categorical sense.
The complex $\Omega^\Phi_{X/k}$ is called the diffracted Hodge complex of $X$ (relative to $\tilde{X}$), and $\Theta$ the Sen operator.

Let $d = \dim(X)$. As, by (ii) and (iii), $\prod_{0 \leq i \leq d}(\Theta + i)$ is nilpotent, we get a decomposition into generalized eigenspaces:

Corollary. Under the assumptions of th. 1, there is a canonically defined endomorphism $\Theta$ of $F_*\Omega^\bullet_{X/k}$, and a canonical decomposition

$$F_*\Omega^\bullet_{X/k} = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} (F_*\Omega^\bullet_{X/k})_i$$

stable under $\Theta$, such that, for all $i \in \mathbb{Z}/p\mathbb{Z}$,

$$\Theta | (F_*\Omega^\bullet_{X/k})_i = -i + \Theta_i,$$

where $\Theta_i$ is a nilpotent endomorphism of $(F_*\Omega^\bullet_{X/k})_i$. 
In particular, for any $a \in \mathbb{Z}$,

\[(0.2) \quad \varepsilon : \varphi^* \Omega^{\phi}_{X/k} \sim \to F_* \Omega^\bullet_{X/k},\]

induces a canonical decomposition

\[(0.3) \quad \bigoplus_{a \leq i < a+p} \Omega^i_{X'/k} [-i] \sim \to \tau^{[a,a+p-1]} F_* \Omega^\bullet_{X/k},\]

a refinement of (0.1) and of a result of Achinger-Suh [A].

**Remarks.** (1) It can be shown [LM] that, for $a = 0$, the decomposition (0.3) coincides with that of [DI] (see end of section 6).

(2) Petrov [P] has constructed an example of a smooth $X/k$, lifted to $W(k)$, such that $\Theta_0 \neq 0$.

(3) Suppose $X$ admits a smooth formal lift $Z/W(k)$ with $Z \otimes W_2(k) = \tilde{X}$. Then $\Omega^{\phi}_{X/k} = \Omega^{\phi}_{Z/W(k)} \otimes^L k$, where $\Omega^{\phi}_{Z/W(k)}$ is a certain perfect complex of $\mathcal{O}_{Z}$-modules, again called a diffracted Hodge complex, and $\Theta$ is induced by an endomorphism $\Theta$ of $\Omega^{\phi}_{Z/W(k)}$, inducing $-i$ on $H^i(\Omega^{\phi}_{Z/W(k)}) = \Omega^i_{Z/W(k)}$. 
1. Strategy

For simplicity and convenience of references to [BL] we’ll work with liftings \( \mathcal{W}(k) \) rather than \( \mathcal{W}_2(k) \) (adaptation to \( \mathcal{W}_2(k) \) easy but technical, see [BL1, 5.16] and remarks at the end of section 5).

Change notation. Fix (formal smooth) \( X/\mathcal{W}(k) \) lifting \( Y := X \otimes_{\mathcal{W}(k)} k \).

Main steps

(a) First assume \( X \) affine, \( X = \text{Spf}(R) \). Using Hodge-Tate cohomology of \( X \) with respect to all (bounded) prisms over \( \text{Spf}(\mathcal{W}(k)) \) (forming the so-called absolute prismatic site \( \text{Spf}(\mathcal{W}(k))_\Delta \) of \( \text{Spf}(\mathcal{W}(k)) \)), construct an absolute Hodge-Tate crystal \( C_R \) in \( p \)-complete complexes over \( \text{Spf}(\mathcal{W}(k))_\Delta \).
A new input (Bhatt-Lurie, Drinfeld): the Cartier-Witt stack \( \text{WCart} \) and the Hodge-Tate stack over \( \text{Spf}(W(k)) \):

\[
\text{WCart}_{\text{Spf}(W(k))}^{\text{HT}} \hookrightarrow \text{WCart}
\]

(an effective Cartier divisor in the Cartier-Witt stack). The universal property of Witt rings with respect to \( \delta \)-structures implies that \( \text{WCart} \) plays the role of an attractor for absolute prisms over \( \text{Spf}(W(k)) \).

Using this, interpret \((p\text{-complete})\) absolute Hodge-Tate crystals on \( \text{Spf}(W(k)) \) as \((p\text{-complete})\) complexes on the Hodge-Tate stack.

Then realize the Hodge-Tate stack as the classifying stack \( BG \) of a certain commutative affine group scheme \( G \) over \( \text{Spf}(W(k)) \) containing \( \mu_p \), and whose quotient \( G/\mu_p \) is isomorphic to \( G_a^\# \), the PD-hull of \( G_a \) at the origin.
(c) Interpret \((p\text{-complete})\) complexes on \(BG\) as \((p\text{-complete})\) complexes \(K\) of \(W(k)\)-modules, endowed with a certain endomorphism \(\Theta\), called the \textbf{Sen operator}, such that \(\Theta^p - \Theta\) is nilpotent on each \(H^i(K \otimes^L k)\). In particular, the Hodge-Tate crystal \(C_R\) above can be described by a \(p\)-complete object

\[
\Omega^{\phi}_{R/W(k)}
\]

of \(D(W(k))\), endowed with a Sen operator \(\Theta\). Using the \textbf{Hodge-Tate comparison theorem}, promote \(\Omega^{\phi}_{R/W(k)}\) to a perfect complex in \(D(R)\), endowed with a multiplicative, increasing filtration \(\text{Fil}^{\text{conj}}\), with \(\text{gr}_i \xrightarrow{\sim} \Omega^i_{R/W(k)}[-i]\), and \(\Theta\) acting on \(\text{gr}_i\) by \(-i\).
(d) For a general formal smooth $f : X \rightarrow \text{Spf}(W(k))$, pasting the
$\Omega^\mathcal{D}_{R/W(k)}$ for the various affine opens $U = \text{Spf}(R)$ of $X$ gives a
filtered perfect complex of $\mathcal{O}_X$-modules

$$\Omega^\mathcal{D}_{X/W(k)}$$

called the diffracted Hodge complex of $X/W(k)$ (a twisted form of
the Hodge complex $\bigoplus \Omega^i_{X/W(k)}[−i]$), with $\text{gr}_i \sim \Omega^i_{X/W(k)}[−i]$, and
equipped with a Sen operator $\Theta$, acting by $−i$ on $H^i(\Omega^\mathcal{D}_{X/W(k)})$. 
(e) Show that the object of $D(X, \mathcal{O}_X)$ underlying $\Omega_{X/W(k)}^\phi$ is the relative Hodge-Tate cohomology complex $\overline{\Delta}_{X/A}$ of $X$ over a special prism $(A, I)$ with $A/I = W(k)$, deduced from the $q$-de Rham prism (by taking invariants under $F_p^*$ and base changing to $W(k)$).

Finally, using this and the prismatic de Rham comparison theorem for $\overline{\Delta}_{X/A}$, construct the desired isomorphism

\begin{equation}
\Omega_{Y'/k}^\phi \cong F^* \Omega_{Y/k}^\bullet
\end{equation}

inducing $C^{-1}$ on $H^i$, where

$$\Omega_{Y/k}^\phi := \Omega_{X/W(k)}^\phi \otimes W(k)$$
2. Prismatic and Hodge-Tate cohomology complexes

Recall: **Prism**: $(A, I, \delta)$:

- $\delta : A \to A$: a delta structure; $x \mapsto \varphi(x) := x^p + p\delta(x)$ lifts Frobenius

\[(\delta \iff \text{(section of } W_2(A) \to A) \iff \text{(lift of } F : A/L^p \to A/L^p))\]

- $I \subset A$: a Cartier divisor; $A$: derived $(p, I)$-complete (i.e.,
  \[A \xrightarrow{\sim} R \lim A \left[ \begin{array}{c} t_1 \to p^n, t_2 \to d^n \end{array} \right] \otimes_{Z[t_1, t_2]}^L Z \text{ if } I = (d)\])

- $p \in I + \varphi(I)A \iff I$ locally generated by distinguished $d$, i.e.
  \[\delta(d) \in A^*\]

Map of prisms: $(A, I) \to (B, J)$; $J = IB$ automatic
Examples: $(\mathbb{Z}_p, (p), \varphi(x) = x)$ (the crystalline prism);
$(\mathbb{Z}_p[[u]], (u - p), \varphi(u) = u^p)$ (a Breuil-Kisin prism);
$(\mathbb{Z}_p[[q - 1]], ([p]_q := 1 + q + \cdots + q^{p-1}), \varphi(q) = q^p)$ (the $q$-de Rham prism)

(Relative) prismatic site [BS, 4.1]:

For $(A, I)$ bounded, $X/(A/I)$ smooth formal, prismatic site $(X/A)_\Delta$:

Objects:

$$\text{Spf}(B/IB) \rightarrow \text{Spf}(B),$$

with $(B, IB)$ bounded.

Maps: obvious

Covers: $(B, IB) \rightarrow (C, IC)$ faithfully flat (i.e., $C (p, IB)$-completely faithfully flat over $B$, [BS, 1.2])
Structure sheaf $\mathcal{O}_\Delta: (X \leftarrow \text{Spf}(B/I) \rightarrow \text{Spf}(B)) \mapsto B$;

Hodge-Tate sheaf $\overline{\mathcal{O}}: (\_ \mapsto B/IB)$; both with $\varphi$-actions

Let $(A, I)$ a bounded prism, set $\overline{A} := A/I$. Fix $X/\overline{A}$ smooth formal.

Prismatic cohomology complex [BS, 4.2]:

$$\Delta_{X/A} := R\nu_*\mathcal{O}_\Delta \in D(X_{et}, A)$$

where $\nu$ is the canonical map of topoi

$$\nu: (\overline{X/A})_\Delta \rightarrow \overline{X}_{et}.$$ 

$$R\Gamma_\Delta(X/A) := R\Gamma((X/A)_\Delta, \mathcal{O}) = R\Gamma(X_{et}, R\nu_*\mathcal{O}_\Delta) \in D(A).$$

Hodge-Tate cohomology complex [BS, 4.2]

$$\overline{\Delta}_{X/A} := R\nu_*\overline{\mathcal{O}}_\Delta = \Delta_{X/A} \otimes_A^L \overline{A} \in D(X_{et}, \mathcal{O}_X).$$
The Hodge-Tate comparison theorem [BS 4.11] gives an $\mathcal{O}_X$-linear isomorphism

$$(HT) \quad \Omega^i_{\mathcal{X}/\mathcal{A}} \otimes (I/I^2)^{-i} \sim H^i(\overline{\Delta}_{\mathcal{X}/\mathcal{A}}),$$

hence $\overline{\Delta}_{\mathcal{X}/\mathcal{A}}$ is a perfect complex of $\mathcal{O}_X$-modules (with $\varphi$-action). On the other hand, the de Rham comparison theorem [BS 15.4] gives a $\varphi$-linear isomorphism (in $D(\mathcal{X}, \overline{\mathcal{A}})$)

$$(dR) \quad \varphi^*_\mathcal{A}\Delta_{\mathcal{X}/\mathcal{A}} \otimes^L \overline{\mathcal{A}} (\sim \varphi^*_\mathcal{A}\overline{\Delta}_{\mathcal{X}/\mathcal{A}}) \sim \Omega^\bullet_{\mathcal{X}/\overline{\mathcal{A}}}. $$
Absolute prismatic site [BL, 4.4.27]:

For a $p$-adic formal scheme $T/\mathbb{Z}_p$, define the absolute prismatic site of $T$,

$$T_\Delta,$$

as the category of maps

$$T \xleftarrow{\text{a}} \text{Spf}(\bar{A}) \to \text{Spf}(A)$$

where $(A, I), \bar{A} = A/I$, runs through all bounded prisms, with the obvious maps, and the topology given by maps with $(A, I) \to (B, IB)$ faithfully flat.

Structural sheaves: $\mathcal{O}_\Delta$ (or $\mathcal{O}$): $(A, I) \mapsto A, \overline{\mathcal{O}}_\Delta$ (or $\overline{\mathcal{O}}$): $(A, I) \mapsto A/I$

Example: $\text{Spf}(\mathbb{Z}_p)_\Delta$ is the category of all bounded prisms.
Definition. \( \hat{D}(A) := \) full subcategory of \( D(A) \) consisting of \((p, I)\)-complete objects.

Definitions. (1) A \( p \)-complete prismatic crystal on \( T_\Delta \) is an object \( E \) of \( D(T_\Delta, \mathcal{O}) \) such that, for all \( A = (A, I, a) \in T_\Delta \), \( E(A) \in \hat{D}(A) \), and for all maps \( (A, I, a) \to (B, IB, b) \), the induced map

\[
B \otimes^L_A E(A) \to E(B)
\]

is an isomorphism. Denote by

\[
\hat{D}_{crys}(T_\Delta, \mathcal{O})
\]

the full subcategory of \( D(T_\Delta, \mathcal{O}) \) consisting of \( p \)-complete prismatic crystals.
(2) A \( p \)-complete Hodge-Tate crystal on \( T_\Delta \) is an object \( E \) of \( D(T_\Delta, \overline{\mathcal{O}}) \) such that, for all \( A = (A, l, a) \in T_\Delta \), \( E(A) \in \hat{D}(\overline{A}) \), and for all maps \( (A, l, a) \to (B, lB, b) \), the induced map

\[
\overline{B} \otimes^L_A E(A) \to E(\overline{B})
\]

is an isomorphism. Denote by

\[
\hat{D}_{\text{crys}}(T_\Delta, \overline{\mathcal{O}})
\]

the full subcategory of \( D(T_\Delta, \overline{\mathcal{O}}) \) consisting of \( p \)-complete Hodge-Tate crystals.
Examples. Let $f : X \to \text{Spf}(\mathcal{W}(k))$ be formal smooth. Then:

(1) $Rf_*\mathcal{O}_\Delta$, 

$(A, I, a) \mapsto (Rf_*\mathcal{O}_\Delta)_{(A, I)} = R\Gamma_\Delta(X_{\overline{A}}/A) := R\Gamma((X_{\overline{A}}/A)\Delta, \mathcal{O}_\Delta) \in \hat{D}(A)$

belongs to $\hat{D}_{\text{crys}}(\text{Spf}(\mathcal{W}(k))\Delta, \mathcal{O})$

(2) $Rf_*\overline{\mathcal{O}}_\Delta$, 

$(A, I, a) \mapsto (Rf_*\overline{\mathcal{O}}_\Delta)_{(A, I)} = R\Gamma_{\Delta}(X_{\overline{A}}/A) := R\Gamma((X_{\overline{A}}/A)\Delta, \overline{\mathcal{O}}_\Delta) \in \hat{D}(\overline{A})$

belongs to $\hat{D}_{\text{crys}}(\text{Spf}(\mathcal{W}(k))\Delta, \overline{\mathcal{O}})$
3. A strange attractor: the Cartier-Witt stack

A priori, the category $\text{Spf}(\mathbb{Z}_p)_\Delta$ of all bounded prisms looks chaotic. However, these prisms are in a kind of magnetic field: they are all attracted by a single big formal stack, the Cartier-Witt stack. The local symmetries of this stack, and of its companion the Hodge-Tate stack, yield the hidden structure on $\Omega^\bullet_{\mathcal{Y}/k}$ described above.

Definition [BL, 3.1.1]. A generalized Cartier divisor on a scheme $\mathcal{X}$ is a pair $(\mathcal{I}, \alpha)$, where $\mathcal{I}$ is an invertible $\mathcal{O}_\mathcal{X}$-module and $\alpha : \mathcal{I} \rightarrow \mathcal{O}_\mathcal{X}$ a morphism of $\mathcal{O}_\mathcal{X}$-modules. When $\mathcal{X} = \text{Spec}(A)$, we identify generalized Cartier divisors on $\mathcal{X}$ with pairs $(I, \alpha)$, where $I$ is an invertible $A$-module and $\alpha : I \rightarrow A$ an $A$-linear map. Morphisms are defined in the obvious way.
Remarks

1. This notion, under the name of “divisor”, was introduced by Deligne in 1988\(^2\). A similar notion was independently devised by Faltings at about the same time [F].

2. A generalized Cartier divisor on \(X\) generates (corresponds to) a log structure \(M_X\) on \(X\) called a Deligne-Faltings log structure of rank 1. In the early 2000’s Lafforgue observed that such an \(M_X\) corresponds to a morphism \(X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]\). That triggered Olsson’s work [Ol].

3. A generalized Cartier divisor \((I, \alpha)\) on \(A\) defines a quasi-ideal in \(A\) in the sense of Drinfeld [Dr], i.e. a differential graded algebra \((I \xrightarrow{\alpha} A)\) concentrated in degree 0 and \(-1\), hence an animated ring \([I \xrightarrow{\alpha} A]\) (object of the derived category of simplicial (commutative) rings).

\(^2\)Letter to L. Illusie, June 1, 1988
The Cartier-Witt stack (Drinfeld’s $\Sigma$) is the formal stack over $\mathbb{Z}_p$

$$\text{WCart} := [\text{WCart}_0/\mathcal{W}^\times]$$

where:

$\mathcal{W} := (p$-typical) Witt scheme over $\mathbb{Z}_p$

$\text{WCart}_0 :=$ formal completion of $\mathcal{W}$ along locally closed subscheme defined by $p = x_0 = 0, \ x_1 \neq 0$, the formal scheme of primitive Witt vectors:

$$\text{WCart}_0 = \text{Spf}(A^0) \quad A^0 := \mathbb{Z}_p[[x_0]][x_1, x_1^{-1}, x_2, x_3, \cdots ]$$

where hat means $p$-completion (and the ring structure on $A^0$ is given by the Witt polynomials).
For a $p$-nilpotent ring $R$, $\text{WCart}_0(R)$ is the set of $a = (a_0, a_1, \cdots) \in W(R)$ with $a_0$ nilpotent and $a_1$ invertible ($\iff \delta(a)$ invertible$^3$).

$W^\times \subset W := Z_p$-group scheme of units in $W$, acting on $\text{WCart}_0$ by multiplication.

$^3$NB. $Fa = (a_0^p + pa_1, \cdots)$, $a^p = (a_0^p, \cdots)$, $\delta(a) = (a_1, \cdots)$. 
For a ring $R$,

$$\text{WCart}(R)$$

is defined as the empty category if $p$ is not nilpotent in $R$, and, if $R$ is $p$-nilpotent, is the groupoid whose objects are Cartier-Witt divisors on $R$,

$$(I \xrightarrow{\alpha} W(R))$$

i.e., generalized Cartier divisors on $W(R)$ such that (Zariski locally over $\text{Spec}(R)$) $\alpha$ maps $I$ to $\text{WCart}_0(R)$ ($\Leftrightarrow \text{Im}(I \to W(R) \to R)$ is nilpotent and $\text{Im}(I \to W(R) \xrightarrow{\delta} W(R))$ generates the unit ideal).
The attracting property of WCart

Let \((A, I)\) be a bounded\(^4\) prism. Then the formal scheme \(\text{Spf}(A)\) (with the \((p, I)\)-adic topology) canonically maps to the formal stack WCart by a map

\[ \rho_{(A, I)} : \text{Spf}(A) \rightarrow \text{WCart} \]

defined as follows. For a point \(f : A \rightarrow R\) of \(\text{Spf}(A)\) with value in a \((p\text{-nilpotent})\) ring \(R\), \(f\) uniquely lifts to a \(\delta\)-map \(f : A \rightarrow W(R)\), by which the inclusion \(I \subset A\) induces a generalized Cartier divisor

\[ \rho_{(A, I)}(f) = (I \otimes_A W(R) \xrightarrow{\alpha} W(R)) \in \text{WCart}(R). \]

Then:\[ f \mapsto \rho_{(A, I)}(f)\] defines \(\rho_{(A, I)}\).

\(^4\)This ensures that \(A\) is classically \((p, I)\)-complete.
Stacky description of prismatic crystals

The ($\infty$-)category of quasi-coherent complexes on $\mathcal{W}$Cart,

$$D(\mathcal{W}\text{Cart}) := \lim_{\mathcal{Spec}(R) \to \mathcal{W}\text{Cart}} D(R)$$

(sometimes denoted $D_{qc}(\mathcal{W}\text{Cart})$) is by definition the inverse limit of the categories $D(R)$ indexed by the category of points of $\mathcal{W}$Cart, i.e. objects $\mathcal{F}$ are coherent rules

$$( (I \to \mathcal{W}(R)) \in \mathcal{W}\text{Cart}(R)) \mapsto \mathcal{F}( (I \to \mathcal{W}(R)) \in D(R).$$

For a bounded prism $(A, I)$, $\rho_{(A, I)} : \text{Spf}(A) \to \mathcal{W}\text{Cart}$, induces a pull-back map

$$\rho^*_{(A, I)} : D(\mathcal{W}\text{Cart}) \to D(\text{Spf}(A)) = \hat{D}(A).$$

For variable $(A, I)$ these maps define a functor

$$D(\mathcal{W}\text{Cart}) \to \hat{D}_{\text{crys}}(\text{Spf}(\mathbb{Z}_p)_{\Delta}, \mathcal{O})$$

where the right-hand side is the category of prismatic crystals on the absolute prismatic site $\text{Spf}(\mathbb{Z}_p)_{\Delta}$ (BL, 3.3.5).
Theorem 2 (BL, Prop. 3.3.5). The functor

$$D(W\text{Cart}) \to \hat{D}_{\text{crys}}(\text{Spf}(\mathbb{Z}_p)_\Delta, \mathcal{O})$$

is an equivalence.

In a sense, the Cartier-Witt attractor plays the role of a final object for the site $\text{Spf}(\mathbb{Z}_p)_\Delta$.

Proof. Use the prism $(A^0, I_0 := (x_0))$, where $A^0$ is the coordinate ring of $W\text{Cart}_0$, and the Zariski cover $\text{Spf}(A^0) \to W\text{Cart}$. 

More generally, for any (bounded) $p$-adic formal scheme $X$, there is defined a formal stack

$$\text{WCart}_X$$

over $\mathbb{Z}_p$, called the Cartier-Witt stack of $X$, which depends functorially on $X$. For $X = \text{Spf}(\mathcal{W}(k))$ as above,

$$\text{WCart}_{\text{Spf}(\mathcal{W}(k))} = \text{Spf}(\mathcal{W}(k)) \times_{\text{Spf}(\mathbb{Z}_p)} \text{WCart}$$

and the analogue of Th. 2 holds.
For $R$ a $p$-nilpotent ring, $\text{WCart}_X(R)$ is the groupoid

$$
\text{WCart}_X(R) = \{(I \overset{\alpha}{\to} W(R)), x \in X(W(R)/^{L}I)\},
$$

where $(I \overset{\alpha}{\to} W(R))$ is a Cartier-Witt divisor on $R$, and $x$ a point of $X$ with value in the animated ring $W(R)/^{L}I$ defined by $(I \overset{\alpha}{\to} W(R))$.

**Examples.**

- $\text{WCart}_{\text{Spf}(\mathbb{Z}_p)} = \text{WCart}$
- $\text{WCart}_{\text{Spec}(k)} = \text{Spf}(W(k))$.

The construction $X \mapsto \text{WCart}_X$ leads to a theory of **prismatization**, developed in [BL1].
4. The Hodge-Tate stack

The category $D(\text{WCart})$ is hard to describe "concretely", but it turns out that $\text{WCart}$ contains an effective Cartier divisor, the 

Hodge-Tate divisor $\text{WCart}^\text{HT}$, whose category of quasi-coherent objects on it has a simple description.
The Hodge-Tate divisor

Definition. The Hodge-Tate divisor is the closed substack

\[ \text{WCart}^{HT} \hookrightarrow \text{WCart}, \]

whose \( R \)-points consist of Cartier-Witt divisors \( I \xrightarrow{\alpha} W(R) \) such that the composition \( I \xrightarrow{\alpha} W(R) \to R \) is zero.

In other words, it’s the fibre product

\[ \text{WCart}^{HT} \to \text{WCart} = [\text{WCart}_0/\mathcal{W}^*] \]

\[ B\mathbb{G}_m = [{\{0\}}/\mathbb{G}_m] \to [{\hat{A}}^1/\mathbb{G}_m] \]

where the right vertical map is induced by the projection \((a_0, a_1, \cdots) \mapsto a_0\).
Equivalently,

\[ \text{WCart}^{\text{HT}} \xrightarrow{\sim} [VW^*/W^*]. \]

Thus, \( R \mapsto (V(1) \in VW^*(R)) \) yields a canonical point, the Hodge-Tate point,

\[ \eta := V(1) : \text{Spf}(\mathbb{Z}_p) \to \text{WCart}^{\text{HT}}. \]

**Theorem 3 ([Dr], [BL]).** The Hodge-Tate point is a flat cover and induces an isomorphism

\[ BW^*[F] := [\text{Spf}(\mathbb{Z}_p)/W^*[F]] \xrightarrow{\sim} \text{WCart}^{\text{HT}}, \]

where \( W^*[F] \) is the stabilizer of \( \eta \), i.e., the group scheme (over \( \mathbb{Z}_p \))

\[ W^*[F] := \text{Ker}(F : W^* \to W^*). \]

**Proof.** By \( xVy = V(Fx.y) \) and faithful flatness of \( F : W^* \to W^* \), \( VW^* = W^*.V(1) \), hence

\[ \text{WCart}^{\text{HT}} = \text{Cone}(W^* \xrightarrow{F} W^*) = BW^*[F]. \]

The left vertical map in above cartesian square is thus identified to

\[ BW^*[F] \to B\mathbb{G}_m \]
Main results ([BL], [Dr])

• Identification of Hodge-Tate crystals with quasi-coherent complexes on $\text{WCart}^{\text{HT}}$

• Identification of $\mathcal{W}^*[F]$ with $G_m^\#$, PD-envelope at 1 of $G_m$, and description as an extension

$$0 \rightarrow \mu_p \rightarrow G_m^\# \rightarrow G_a^\# \rightarrow 0,$$

where $G_a^\# = \text{PD-envelope of } G_a$ at 0

• Identification of $D(\text{WCart}^{\text{HT}})$ with the category of Sen complexes, i.e. the full subcategory of the $(\infty)$-category $\hat{D}(\mathbb{Z}_p[\Theta])$ of objects $M$ of $\hat{D}(\mathbb{Z}_p)$ endowed with an endomorphism $\Theta$ such that $\Theta$ (the Sen operator) has the property that $\Theta^p - \Theta$ on $H^*(M \otimes^L \mathbf{F}_p)$ is locally nilpotent.

(Discussed in Section 5.)
• Identification of Hodge-Tate crystals with quasi-coherent complexes on $\text{WCart}^{\text{HT}}$

Similarly to $D(\text{WCart})$ define

$$D(\text{WCart}^{\text{HT}}) := \lim_{\text{Spec}(R) \to \text{WCart}^{\text{HT}}} D(R).$$

If $(A, I)$ is a bounded prism, then $\rho(A, I) = \text{Spf}(A) \to \text{WCart}$ restricts to

$$\rho^{\text{HT}}_{(A, I)} : \text{Spf}(A/I) \to \text{WCart}^{\text{HT}},$$

$$\rho(A, I)(f : A/I \to R) = (I \otimes_A W(R) \overset{\alpha}{\to} W(R)) \in \text{WCart}^{\text{HT}}(R).$$
From
\[ \rho_{(A,I)}^{\text{HT}} : \text{Spf}(A/I) \to \text{WCart}^{\text{HT}}, \]
get pull-back map
\[ (\rho_{(A,I)}^{\text{HT}})^* : D(\text{WCart}^{\text{HT}}) \to \hat{D}(A/I), \]
and functor
\[ D(\text{WCart}^{\text{HT}}) \to \varprojlim_{(A,I)} \hat{D}(A/I) = \hat{\text{D}}_{\text{crys}}(\text{Spf}(\mathbb{Z}_p)_{\Delta}, \mathcal{O}), \]

where the right hand side is the (\(\infty\)-)category of \(p\)-complete Hodge-Tate crystals (section 2, end). The above classification of \(p\)-complete crystals, restricted to Hodge-Tate crystals, yields:

**Theorem 4.** The above functor is an equivalence:
\[ D(\text{WCart}^{\text{HT}}) \sim \hat{\text{D}}_{\text{crys}}(\text{Spf}(\mathbb{Z}_p)_{\Delta}, \mathcal{O}) \]
More generally, given a $p$-adic formal scheme $X$, one defines the Hodge-Tate divisor $\text{WCart}_{X}^{\text{HT}}$ by the pull-back square:

\[
\begin{array}{ccc}
\text{WCart}_{X}^{\text{HT}} & \longrightarrow & \text{WCart}_{X} \\
\downarrow & & \downarrow \\
\text{WCart}^{\text{HT}} & \longrightarrow & \text{WCart}
\end{array}
\]

**Examples.** For $k$ as above,

\[
\text{WCart}_{\text{Spec}(k)}^{\text{HT}} = \text{Spec}(k),
\]

\[
\text{WCart}_{\text{Spf}(W(k))}^{\text{HT}} = \text{Spf}(W(k)) \times_{\text{Spf}(\mathbb{Z}_p)} \text{WCart}^{\text{HT}}.
\]

and we have the equivalence

\[
D(\text{WCart}_{\text{Spf}(W(k))}^{\text{HT}}) \sim \widehat{D}_{\text{crys}}(\text{Spf}(W(k))_{\Delta}, \overline{\mathcal{O}}),
\]
Corollary. There are canonical equivalences

\[ \hat{D}_{\text{crys}}(\text{Spf}(W(k)))_{\Delta, \overline{\mathcal{O}}} \xleftarrow{\sim} D(W\text{Cart}^{\text{HT}}_{\text{Spf}(W(k))}) \xrightarrow{\eta^*} D((BW^*[F])_{\text{Spf}(W(k))}), \]

\[ \mathcal{E} \mapsto \mathcal{E}_{\eta} := \eta^*(\mathcal{E}) \]

where

- \( \eta = V(1) : \text{Spf}(W(k)) \to W\text{Cart}^{\text{HT}}_{\text{Spf}(W(k))} \) is the canonical point defined above

- a Hodge-Tate crystal is identified with the corresponding quasi-coherent complex on the Hodge-Tate stack.

- an object \( \mathcal{E} \) of \( D(W\text{Cart}^{\text{HT}}_{\text{Spf}(W(k))}) = D((BW^*[F])_{\text{Spf}(W(k))}) \) is identified by \( \mathcal{E} \mapsto \mathcal{E}_{\eta} := \eta^*(\mathcal{E}) \) with a pair of an object \( E \in \hat{D}(\text{Spf}(W(k))) \) and an action \( \alpha \) of \( G^\#_m \) on it.
The group scheme $G^\#_m$

**Proposition** ([Dr, 3.2.6], [BL, 3.4.11, 3.5.18]) (i) The composite $W[F] \rightarrow W \rightarrow G_a$ induces an isomorphism

$$W[F] \sim \rightarrow G^\#_a = \text{Spec}(D(t)\mathbb{Z}_p[t]) = \text{Spec}(\Gamma_{\mathbb{Z}_p}(\mathbb{Z}_pt)).$$

(ii) The composite $W^*[F] \rightarrow W^* \rightarrow G_m$ induces an isomorphism

$$W^*[F] \sim \rightarrow G^\#_m = \text{Spec}(D(t^{-1})\mathbb{Z}_p[t, t^{-1}]).$$

(iii) There is an exact sequence of group schemes (over $\mathbb{Z}_p$)

$$0 \rightarrow \mu_p \xrightarrow{[\cdot]} G^\#_m \xrightarrow{\log(\cdot)} G^\#_a \rightarrow 0,$$

split over $\mathbb{F}_p$, where

$$G^\#_a := \text{Spec}(\mathbb{Z}_p\langle t \rangle)(\sim \rightarrow W[F])$$

is the PD-envelope of $G_a$ at the origin.
Proof. Main point is (i). Drinfeld’s argument: use description of $\mathbb{Z}_p\langle t \rangle$ by generators $u_n = t^{p^n}/p \frac{p^n-1}{p-1}$ and relations $u_n^p = pu_{n+1}$, and Joyal’s theorem to the effect that the coordinate ring $B = \Gamma(W, O)$ of $W$ is the free $\delta$-ring on one indeterminate $y_0$, i.e., is the polynomial ring

$$B = \mathbb{Z}_p[y_0, y_1, \cdots],$$

with $y_n = \delta^n(y_0)$. 
5. Sen operators, Hodge diffraction

Let $\mathcal{E} \in D(W\text{Cart}^{HT}) = D(BG_m^\#)^5$, that we identify with the pair of an object $E = \mathcal{E}_\eta \in \hat{D}(\text{Spf}(W(k)))$ and an action $\alpha : G_m^\# \to \text{Aut}(E)$. Consider the induced infinitesimal action

$$\text{Lie}(\alpha) : \text{Lie}(G_m^\#) \to \text{End}(E),$$

where $\text{Lie}(G_m^\#) = G_m^\#(\text{Spf}(W(k))[\varepsilon]/(\varepsilon^2))$.

In particular, the point $1 + [\varepsilon] \in \text{Lie}(G_m^\#)$ gives an endomorphism

$$\Theta_\mathcal{E} \in \text{End}(E)$$

called the Sen operator.

The Sen operators satisfy a Leibniz rule

$$\Theta_{\mathcal{E} \otimes \mathcal{F}} = \Theta_\mathcal{E} \otimes \text{Id}_\mathcal{F} + \text{Id}_\mathcal{E} \otimes \Theta_\mathcal{F}.$$  

\footnote{In this section we work over $W(k)$ and in general omit the subscript $\text{Spf}(W(k))$.}
Examples. (1) $\Theta_{\mathcal{O}_{\text{WCartHT}}} = 0$.

(2) For the (Hodge-Tate) Breuil-Kisin twist $\mathcal{O}_{\text{WCartHT}}\{1\}$, i.e., the line bundle on $\text{WCart}^\text{HT}$ defined by $I/I^2$, where $I : (A, I) \mapsto I$:

$$\Theta_{\mathcal{O}_{\text{WCartHT}}} \{1\} = \text{Id}.$$

Hence $\Theta_{\mathcal{O}_{\text{WCartHT}}} \{n\} = n\text{Id}$.

(Note: The Hodge-Tate crystal $\mathcal{O}_{\text{WCartHT}}\{1\}$ is induced on the Hodge-Tate divisor from the (crystalline) Breuil-Kisin line bundle $\mathcal{O}_{\text{WCart}}\{1\}$, a prismatic $F$-crystal [BL, 3.3.8] satisfying $\varphi^* \mathcal{O}_{\text{WCart}}\{1\} \sim I^{-1} \otimes \mathcal{O}_{\text{WCart}}\{1\}$).
(3) The cartesian square

\[
\begin{array}{ccc}
G_m^\# & \rightarrow & \text{Spf}(W(k)) \\
\downarrow & & \downarrow \eta \\
\text{Spf}(W(k)) & \rightarrow & \text{WCart}^{\text{HT}}
\end{array}
\]

yields

\[\eta^* \eta_* \mathcal{O} = \hat{\mathcal{O}}^\#_{G_m},\]

where the right hand side denotes the \(p\)-completion of the coordinate ring \(D_{(t-1)}(W(k)[t, t^{-1}])\) of \(G_m^\#\). One has:

\[\Theta_{\hat{\mathcal{O}}^\#_{G_m}} = t \partial / \partial t.\]
Denote by
\[ \hat{D}(W(k)[\Theta]) \]
the category of pairs \((M, \Theta_M)\) where \(M\) is a \(p\)-complete object of \(D(W(k))\) and \(\Theta_M\) an endomorphism of \(M\).

**Theorem 5 [BL, 3.5.8].** The functor
\[ D(W\text{Cart}^{\text{HT}}_{\text{Spf}(W(k))}) \to \hat{D}(W(k)[\Theta]), \mathcal{E} \mapsto (\mathcal{E}_\eta, \Theta_\mathcal{E}) \]
is fully faithful and its essential image consists of pairs \((M, \Theta_M)\) such that \((\Theta_M)^p - \Theta_M\) is locally nilpotent\(^6\) on \(H^*(M \otimes^L_{W(k)} k)\) (such pairs are called Sen complexes).

\(^6\) i.e., for each \(x \in H^i\), there exists \(n(x)\) such that \((\Theta_M^p - \Theta_M)^n.x = 0\) for \(n \geq n(x)\).
Proof. Main points: (i) A fixed point formula: for $\mathcal{E} \in D(W\text{Cart}^{\text{HT}})$,

$$\mathcal{E} \sim (\eta_* \eta^* \mathcal{E})^{\Theta=0}$$

(ii) Dévissage (using the co-regular representation of $\hat{\mathcal{O}}_{\mathcal{G}_m}^\#$) showing that $D(W\text{Cart}^{\text{HT}})$ is generated, under shifts and colimits, by the Breuil-Kisin twists $\mathcal{O}_{W\text{Cart}^{\text{HT}}} \{n\}$ $(n \geq 0)$. 
The diffracted Hodge complex $\Omega_{\mathcal{X}/W(k)}^\mathcal{D}$

Let’s come back to our formal smooth $f : \mathcal{X} \rightarrow \operatorname{Spf}(W(k))$.

(a) Assume first that $\mathcal{X}$ is affine, $\mathcal{X} = \operatorname{Spf}(R)$.

Denote by

$$\Omega_{R/W(k)}^\mathcal{D} \in \hat{D}(\operatorname{Spf}(W(k))[\Theta])$$

the Sen complex associated with the $p$-complete Hodge-Tate crystal over $\operatorname{Spf}(W(k))$

$$(A, I) \mapsto (Rf_*\overline{\mathcal{O}}_\mathcal{X})(A, I) = R\Gamma_{\mathcal{X}/A}(X_{\overline{A}}/A) := R\Gamma((X_{\overline{A}}/A)_{\mathcal{X}}, \overline{\mathcal{O}}) \in \hat{D}(\overline{A})$$

This Sen complex is called the diffracted Hodge complex of $R/k$.

The canonical truncation filtration of $R\Gamma_{\mathcal{X}/A}(X_{\overline{A}}/A)$ for $(A, I) \in \operatorname{Spf}(W(k))_\mathcal{X}$ defines a canonical increasing, multiplicative filtration of $\Omega_{R/W(k)}^\mathcal{D}$, called the conjugate filtration, which is stable under $\Theta$,

$$\operatorname{Fil}^\mathcal{D}_{\bullet, \Theta} = (\operatorname{Fil}^\mathcal{D}_0 \rightarrow \operatorname{Fil}^\mathcal{D}_1 \rightarrow \cdots).$$
It follows from the Hodge-Tate comparison theorem and the smoothness of $R/W(k)$ that

$$\text{gr}^\text{conj}_i = \Omega^i_{R/W(k)}[-i][-i]$$

In particular, $\text{Fil}_0^{\text{conj}} = R$, so that $\Omega^\Phi_{R/W(k)}$ can be promoted to a filtered object of $\hat{\mathcal{D}}(R)[\Theta]$, which is perfect as a filtered object of $\hat{\mathcal{D}}(R)$.

By Examples (1) and (2) above, we have

$$\Theta|H^i(\Omega^\Phi_{R/W(k)}) = -i.$$
(b) For a general formal smooth $f : X \to \text{Spf}(W(k))$, the $\Omega_{R/W(k)}^\phi$ patch to a filtered perfect complex in $D(X, O_X)$, called the


diffracted Hodge complex of $X/W(k)$

$$\Omega_{X/W(k)}^\phi,$$

equipped with a Sen operator $\Theta$ satifying

$$\Theta|H^i(\Omega_{X/W(k)}^\phi) = -i.$$ 

(which implies (the already known) fact that

$\Theta^p - \Theta$ on $H^*(\Omega_{X/W(k)}^\phi \otimes^L k)$ is nilpotent, and even zero (as $H^*(\Omega_{X/W(k)}^\phi)$ is locally free of finite type over $X$).
Remarks. (1) The Hodge complex $\Omega^*_X/W(k) := \oplus_i \Omega^i_{X/W(k)}[-i]$ and the diffracted one $\Omega^{\mathcal{D}}_X/W(k)$ are both filtered perfect complexes in $D(X, \mathcal{O}_X)$: the former one, with the trivial filtration, with \( \text{gr}^i = \Omega^i_{X/W(k)}[-i] \), the latter one with the canonical filtration, with \( \text{gr}_i = \Omega^i_{X/W(k)}[-i] \{-i\} \) (and the additional structure $\Theta$). Bhatt and Lurie view this deviation and enrichment as a diffraction phenomenon, like a wave being diffracted by a slit ($\eta : \text{Spf}(W(k)) \to W_{\text{Cart}}^{\text{HT}}$).

(2) Let $K := W(k)[1/p]$ and $C := \hat{K}$. It is shown in [BL, 3.9.5, 4.7.22] that by extending the scalars to $\mathcal{O}_C$, and using the prismatic - étale comparison theorem, $\Theta$ corresponds to the classical Sen operator on the (semilinear) representation $C \otimes_{W(k)} H^*(X_K, \mathbb{Z}_p)$ of $\text{Gal}(K/K)$ and (for $X/W(k)$ proper and smooth) yields the Hodge-Tate decomposition

$$C \otimes H^n(X_K, \mathbb{Z}_p) \sim \oplus_i C(-i) \otimes_{W(k)} H^{n-i}(X, \Omega^i_{X/W(k)}).$$
End of proof of Th. 1.

Recall: \( Y := X \otimes_{W(k)} k \). Define

\[
\Omega_{Y/k}^{\Phi} := \Omega_{X/W(k)}^{\Phi} \otimes_{W(k)}^L k \in D(Y, \mathcal{O}_Y),
\]

and let again \( \Theta \) denote the endomorphism induced by the Sen operator of \( \Omega_{X/W(k)}^{\Phi} \).

As we already know that

(i) \( H^i(\Omega_{Y/k}^{\Phi}) \xrightarrow{\sim} \Omega^i_{Y/k} \) canonically,

(ii) \( \Theta \) is a derivation, and acts by \(-i\) on \( H^i \),

it remains to construct the isomorphism (in \( D(Y', \mathcal{O}_{Y'}) \))

(0.2) \[
\varepsilon : \varphi^*\Omega_{Y/k}^{\Phi} \xrightarrow{\sim} F_*\Omega^\bullet_{Y/k},
\]

with the property:

(iii) \( \varepsilon \) is multiplicative and (via (i)) induces the Cartier isomorphism \( C^{-1} : \Omega^i_{Y'/k} \xrightarrow{\sim} H^i(F_*\Omega^\bullet_{Y/k}) \) on \( H^i \).
Interlude: Sen complexes and evaluation of Hodge-Tate crystals

A preliminary is needed for the construction of $\varepsilon$.

Recall that if $\mathcal{E}$ is a (p-complete) Hodge-Tate crystal on $\text{Spf}(\mathcal{W}(k))_\Delta$, the corresponding Sen complex $(E, \Theta)$ is defined by

$$E = \eta^* \mathcal{E},$$

where $\mathcal{E}$ is identified with an object of $D(\text{WCart}^{\text{HT}}) = D(B\mathbb{G}_m^\#)$, and

$$\eta : \text{Spf}(\mathcal{W}(k)) \to B\mathbb{G}_m^\#$$

is the point $V(1)$, corresponding to the trivial $\mathbb{G}_m^\#$-torsor on $\text{Spf}(\mathcal{W}(k))$. 
Let \((A, I) \in \text{Spf}(\mathcal{W}(k))_\Delta\). Consider the canonical map

\[
\rho^\text{HT}_{(A, I)} : \text{Spf}(\overline{A}) \to \text{WCart}^\text{HT} = B\mathbb{G}_m^\# 
\]

It corresponds to a \(\mathbb{G}_m^\#\)-torsor \(\mathcal{P} = \mathcal{P}_{(A, I)}\) over \(\text{Spf}(\overline{A})\), and this torsor is trivial if and only if one can fill in the diagram (of \(A\)-linear maps)

\[
\begin{array}{ccc}
I & \to & \mathcal{W}(\overline{A}) \\
\downarrow & & \downarrow V(1) \\
\mathcal{A} & \longrightarrow & \mathcal{W}(\overline{A}) \\
\downarrow & \psi_{(A, I)} & \\
A & \overset{\psi_{(A, I)}}{\longrightarrow} & \mathcal{W}(\overline{A})
\end{array}
\]

with a top horizontal \(A\)-linear map \(\xi : I \to \mathcal{W}(\overline{A})\) making the square commute, where \(\psi_{(A, I)}\) is the unique lift of \(A \to \overline{A}\) compatible with \(\delta\).
We’ll say that \((A, I)\) is neutral if \(\rho_{(A,I)}^{HT}\) factors through \(\eta\), i.e., \(\mathcal{P}_{(A,I)}\) is trivial. If \((A, I)\) is neutral, then

\[
\mathcal{E}(\overline{A}) \xrightarrow{\sim} \overline{A} \otimes_{\mathcal{W}(k)} \mathcal{E}_\eta
\]

in \(\hat{D}(\overline{A})\).

Consider the \(q\)-de Rham prism

\(Q := (\mathbb{Z}_p[[q - 1]], ([p]_q), \varphi(q) = q^p)\) on which \(i \in \mathbb{F}_p^*\) acts by \(q \mapsto q^{[i]}\) ([\(i\] \(\in\) \(\mathbb{Z}_p^*\) the Teichmüller representative). Let

\[
Q_0 := Q^{\mathbb{F}_p^*}
\]

By [BL, 3.8.6]

\[
Q_0 = (\mathbb{Z}_p[[\tilde{p}]], (\tilde{p}), \varphi(q) = q^p), \ \tilde{p} := \sum_{i \in \mathbb{F}_p} q^{[i]}.
\]

and the prism \((A, I) = \mathcal{W}(k) \otimes_{\mathbb{Z}_p} Q_0\) is neutral.
Remark (Gabber). The element $p - [p] \in W(\mathbb{Z}_p)$ is of the form $Vx$, for $x$ with ghost coordinates

$$w(x) = (1 - p^{p-1}, 1 - p^{p^2-1}, \ldots),$$

and $x$ is in the image of $F$ if and only if $p$ is odd. Therefore the Breuil-Kisin prism $(A, I) = (W(k)[[u]], (p - u), u \mapsto u^p)$ has $A/I = W(k)$, but is neutral if and only if $p$ is odd.
Construction of $\varepsilon$.

Applying the above to the Hodge-Tate crystal $\mathcal{E} = Rf_*\mathcal{O}_\Delta$ for $f : \text{Spf}(R) \to \text{Spf}(W(k))$, and the prism $(A, I) = W(k) \otimes \mathbb{Z}_p Q_0$, with $\overline{A} = W(k)$ we find

$$\Omega^\phi_{R/W(k)} \sim (Rf_*\mathcal{O}_\Delta)(A, I)$$

in $\hat{D}(R)$, and then, for a general formal smooth $X/W(k)$,

$$\Omega^\phi_{X/W(k)} \sim \Delta_X/A$$

in $D(X, \mathcal{O}_X)$. The de Rham comparison theorem (dR) thus provides a multiplicative isomorphism (in $D(X, W(k))$)

$$\varphi_*\Omega^\phi_{X/W(k)} \sim \varphi_*\Omega^\bullet_{X/W(k)},$$

which, by reduction mod $p$ yields the desired isomorphism (in $D(Y', \mathcal{O}_{Y'})$)

$$\varepsilon : \varphi_*\Omega^\phi_{Y/k} \sim F_*\Omega^\bullet_{Y/k}. \quad (0.2)$$
Remarks on the mod $p^2$ lifted case

A formal smooth lifting $X$ of $Y$ over $\text{Spf}(W_2(k))$ instead of over $\text{Spf}(W(k))$ gives rise to a similar story and yields the general case of Th. 1. Note, however, that

$$\text{WCart}^{\text{HT}}_{\text{Spf}(W_n(k))} \not\xrightarrow{\sim} (BG^m_m)_{\text{Spf}(W_n(k))}.$$ 

(e. g., $\text{WCart}^{\text{HT}}_{\text{Spec}(k)} = \text{Spec}(k)$). For all $n \geq 1$,

$$\text{WCart}^{\text{HT}}_{\text{Spf}(W_n(k))} = [\text{Spf}(W_n(k))^\Phi / G^m_m],$$

where $\text{Spf}(W_n(k))^\Phi$ is the diffracted Hodge stack of $\text{Spf}(W_n(k))$, defined by the fiber square

$$\begin{CD}
\text{Spf}(W_n(k))^\Phi @>>> \text{WCart}^{\text{HT}}_{\text{Spf}(W_n(k))} \\
@VVV \quad @VVV \\
\text{Spf}(W(k)) @>{\eta}>> \text{WCart}^{\text{HT}}_{\text{Spf}(W(k))}.
\end{CD}$$
In particular [BL1, 5.15], for $n \geq 2$,

$$\text{WCart}^{\text{HT}}_{\text{Spec}(W_n(k))} \times_{\text{Spf}(W_n(k))} \text{Spec}(k) \xrightarrow{\sim} [G^\#_a/G^\#_m]_{\text{Spec}(k)}.$$

Therefore the composite

$$\text{Spec}(k) \rightarrow \text{Spf}(W(k)) \xrightarrow{\eta} (B G^\#_m)_{\text{Spf}(W(k)}$$

factors through a unique map

$$\eta_2 : \text{Spec}(k) \rightarrow \text{WCart}^{\text{HT}}_{\text{Spec}(W_2(k))} \times_{\text{Spec}(W_2(k))} \text{Spec}(k) = [G^\#_a/G^\#_m]_{\text{Spec}(k)}$$

a section of $[G^\#_a/G^\#_m]_{\text{Spec}(k)}$, whose automorphism group is $(G^\#_m)_{\text{Spec}(k)}$.

This suffices to carry over the arguments to the mod $p^2$ case.
6. An alternate approach: endomorphisms of the de Rham functor (after Li-Mondal, Mondal)

Let $Y/k$ be smooth. The construction of a Sen structure on $F^*\Omega^\bullet_{Y/k}$ provided by a formal smooth $X/W_2(k)$ lifting $Y$ uses the deus ex machina $W_{\text{Cart}}$. One can ask:

(1) Can one understand this hidden structure more concretely?

(2) Can one bypass $W_{\text{Cart}}$ to construct it?

While (1) remains largely open, Li-Mondal [LM] have recently given an independent proof of Th. 1, which doesn’t use prismatication, but instead, a certain ring stack $G_{dR}$ over $W(k)$, the de Rham stack (an avatar of $W_{\text{Cart}}$), which generates the de Rham cohomology functor.

It was subsequently shown by Mondal [M] that this stack is not a deus ex machina, but, in fact, can be reconstructed from the de Rham cohomology functor.
de Rham cohomology functor
(Drinfeld, Li-Mondal, Bhatt) ⇑ ⇓ (Mondal)

The de Rham stack $G_a^{dR}$

⇓ (Li-Mondal)

Endomorphisms of the de Rham functor

⇓

Theorem 1
The de Rham stack

The de Rham stack is the ring stack over $\text{Spf}(\mathcal{W}(k))$

$$G_{a}^{\text{dR}} := [G_{a}/G_{a}^{\#}]$$

where $G_{a}^{\#} = \mathcal{W}[F] = \text{Spec}(\mathcal{W}(k)\langle t \rangle)$ is viewed as a quasi-ideal in $G_{a}$ via the canonical map

$$G_{a}^{\#} \to G_{a}$$

induced by the projection $\mathcal{W} \to G_{a}, \ x \mapsto x_{0}$, corresponding to $\mathcal{W}[t] \to \mathcal{W}\langle t \rangle$.\(^7\) Points of $G_{a}^{\text{dR}}$ with value in a $p$-complete $\mathcal{W}(k)$-algebra $R$ are the groupoid underlying the animated $\mathcal{W}(k)$-algebra

$$G_{a}^{\text{dR}}(R) = (G_{a}^{\#}(R) \to G_{a}(R)).$$

\(^{7}\) (an analogue of the Simpson stack $[G_{a}/\widehat{G_{a}}]$ in characteristic zero)
Relations with $\text{WCart}$ and de Rham cohomology

• Reconstruction of de Rham cohomology

(Bhatt) For $X/\text{Spf}(W(k))$ formal smooth, define the de Rham stack of $X$

$$X^{\text{dR}}$$

by $X^{\text{dR}}(R) = X(G^\text{dR}_a(R))$ on $p$-complete $W(k)$-algebras $R$, i.e., for $X = \text{Spf}(A)$, $X^{\text{dR}}(R) = \text{Hom}(A, G^\text{dR}_a(R))$, Hom taken in the category of animated $W(k)$-algebras.

Theorem 6 (Bhatt, Li-Mondal) There is a functorial isomorphism

$$R\Gamma_{dR}(X/W(k)) = R\Gamma(X^{\text{dR}}, \mathcal{O})$$

The definition of $X^{\text{dR}}$ is a special case of Li-Mondal’s theory of unwinding [LM].

---

8(elaborating on a theorem of Drinfeld [Dr0,Th. 2.4.2])
Relation with $W\text{Cart}$ and the de Rham point

(a) (Drinfeld) $G_{a}^{dR} \sim [W \xrightarrow{p} W]$

(b) Consider the de Rham point

$\rho_{dR} = \rho(z_{p},(p)) : \text{Spf}(Z_{p}) \rightarrow W\text{Cart}$

(corresponding to $p = (p, 1 - p^{-1}, \cdots) \in W\text{Cart}_{0}(Z_{p})$).

By Drinfeld’s formula above $\rho_{dR}$ “generates” the de Rham stack, and, thanks to the prismatic de Rham comparison theorem yields, by pull-back, another proof of Th. 6 [BL, Prop. 5.4.8].
Endomorphisms of the de Rham functor

By unwinding and using that $G_{dR}^a$ is an affine stack in the sense of Toën [T] Li-Mondal [LM] show that $G_{dR}^a$ controls the endomorphisms of the de Rham functor. In particular, they prove:

**Theorem 7** [LM, Th. 4.23] For a $k$-algebra $B$, let $\text{CAlg}(D(B))$ denote the category of commutative algebra objects in the ($\infty$-) category $D(B)$. Consider the group functor on the category of $k$-algebras defined by

$$F : B \mapsto \text{Aut}(\tilde{R} \mapsto \Omega_{\tilde{R} \otimes W_2(k)k/k}^\bullet \otimes_k B \in \text{CAlg}(D(B))),$$

where $\tilde{R}$ runs through the smooth $W_2(k)$-algebras. Then $F$ is represented by $G_{m,k}^\#$. 
Applying Th. 7 for the Hopf algebra $B = \Gamma(G_{m,k}^\#, \mathcal{O})$, Li-Mondal deduce the (functorial in $\tilde{R}$) action of $G_{m,k}^\#$ on $\Omega_{\tilde{R}\otimes k/k}^\bullet$, and, finally, the Sen structure given in Th. 1.

As a bonus, they prove:

**Corollary (1)** There is a unique splitting

$$\mathcal{O}_{\tilde{X}_k} \oplus \Omega^1_{\tilde{X}_k}[-1] \xrightarrow{\sim} \tau^\leq 1F_*\Omega^\bullet_{\tilde{X}_k/k},$$

inducing $C^{-1}$ on $H^i$, and functorial in the smooth scheme $\tilde{X}/W_2(k)$. In particular, the splittings constructed by Drinfeld, Bhatt-Lurie and Li-Mondal coincide.

(2) There is **no** functorial splitting $F_*\Omega^\bullet_{\tilde{X}_k/k} \xrightarrow{\sim} \bigoplus_i H^i(F_*\Omega^\bullet_{\tilde{X}_k/k})[-i]$ as functors to $\text{CAlg}(D(k))$ from smooth schemes $\tilde{X}$ over $W_2(k)$.

**Remark.** Part (2) was proved independently by Mathew.
Reconstruction of the de Rham stack from de Rham cohomology

The functor $R \mapsto \Omega^\bullet_{R/k}$ from the category of smooth $k$-algebras to $\text{CAlg}(D(k))$ extends by left Kan extension to a functor

$$dR : \text{ARings}_k \to \text{CAlg}(D(k)), \ R \mapsto L\Omega^\bullet_{R/k},$$

where $\text{ARings}_k$ is the category of animated $k$-algebras. As $dR$ commutes with colimits, $dR$ has a right adjoint

$$dR^\vee : \text{CAlg}(D(k)) \to \text{ARing}_k.$$

Let $\text{Alg}_k \subset \text{ARings}_k$ be the full subcategory of usual commutative $k$-algebras, and

$$dR^\vee_0 : \text{Alg}_k \to \text{ARing}_k$$

be the restriction of $dR^\vee$ along $\text{Alg}_k \subset \text{ARings}_k \to \text{CAlg}(D(k))$.

**Theorem 8.** (Mondal). There is a canonical isomorphism

$$dR^\vee_0 \sim (G^dR_a)_k.$$
7. Questions

This theory of diffraction and Sen complexes forms a new territory, which has not yet been much explored. Here are a few questions.

**Question 1.** Is there a smooth $Y/k$, liftable to $W_2(k)$, such that

\[(*) \quad F_* \Omega^\bullet_{Y/k} \not\sim \oplus_i H^i(F_* \Omega^\bullet_{Y/k})[-i] \]

in $D(Y', O_{Y'})$?

Question already raised in [DI]. A counterexample should have dimension $\geq p + 1$. By Cor. (2) to Th. 7, there is no splitting (*) of $F_* \Omega^\bullet_{Y/k}$, for $Y = \tilde{Y} \otimes k$, which is *multiplicative* (i.e., with values in $\text{CAlg}(D(k))$) and *functorial* in the lifting $\tilde{Y}/W_2(k)$. 
Questions 2. Let $Y/k$ be smooth, having a lifting $\tilde{Y}$ to $W_2(k)$, so that by Th. 1 we have a Sen structure $(\Omega_{Y/k}^\phi, \Theta, \varepsilon)$ on $F_*\Omega^\bullet_{Y/k}$.

(a) Does there exist a pair $(Y, \tilde{Y})$ such that, for each $i \in \mathbb{Z}/p\mathbb{Z}$, $\Theta_i \in \text{End}((F_*\Omega^\bullet_{Y/k})_i)$ is non-zero?

(Petrov [P] constructed an example with $\Theta_0|\tau^{<p}(F_*\Omega^\bullet_{Y/k})_0$ not 0.)

(b) ([BL, 4.7.20]) Is there a bound, independent of $\dim(Y)$ for the orders of nilpotency of the $\Theta_i$'s?

(c) The isomorphism classes of lifts $\tilde{Y}$ form an affine space $A$ under $\text{Ext}^1(\Omega^1_{Y'/k}, \mathcal{O}_{Y'})$. For each $x \in A$, $\Theta_0(x)$, restricted to $\tau^{<p}(F_*\Omega^\bullet_{Y/k})_0$ is an element $c(x) \in \text{Ext}^p(\Omega^p_{Y'/k}, \mathcal{O}_{Y'})$. Can one explicitly describe the map

$$c : A \to \text{Ext}^p(\Omega^p_{Y'/k}, \mathcal{O}_{Y'})?$$
Question 3. Generalize Sen structures to families, i.e., replace $W_2(k)$ by a parameter space $T$ over $W_2(k)$.

Question 4. (Bhatt) Is there an analogue of the Sen story over other prisms than $(W(k), (p))$? Suppose $(A, I)$ is an absolute (bounded) prism, and $X \to \text{Spf}(A/I)$ formal smooth is lifted to $\tilde{X}$ formal smooth over $\text{Spf}(A)$ (or just $\text{Spf}(A/I^2)$), does the datum of $\tilde{X}$ gives extra structure on $\overline{\Delta}_{X/A} \in D(X, \mathcal{O}_X)$?

Finally, let me mention 3 problems on which there is ongoing work:
(a) Behavior of $\Theta$ with respect to the (decreasing) Hodge filtration\(^9\) of $\Omega^D_{Y/k}$ and analogy of $\Theta^p - \Theta$ with a $p$-curvature. Link with Drinfeld’s $\Sigma'$ [Dr, section 5] and the extended Hodge-Tate stack

$$[G_{dR}^a / G_m]$$

of which $B G_m^\#$ is an open substack. Ongoing work by Bhatt-Lurie [BL 4.7.23].

(b) Problem of reconstructing of $WCart$ from prismatic cohomology: generalization of Th. 8 (reconstruction of $G_{dR}^a$ from de Rham cohomology). Ongoing work by Mondal.

(c) Derived and log variants. Ongoing work by (Mathew-Yao, Mondal).

\(^9\) (deduced from the naive filtration of $F_*\Omega^\bullet_{Y/k}$ by the isomorphism $\Omega^D_{Y'/k} \iso F_*\Omega^\bullet_{Y/k}$)
References


