André's flatness lemma + applications

(joint with P. Scholze)

Fix a prime p

I) The lemma

Def: A perfect prism is a pair (A, I) where

- $A = W(S)$ for a perfect $\mathbb{F}_p$-alg S
- $I = (d) \subset W(S)$ with $d = (d_0, d_1, \dots)$

satisfies
- S is d_0-adically comp
- $d_1 \in S^*$

$$\left\{ \begin{matrix} \text{perfect} \\ \text{prisms} \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{perfectoid} \\ \text{rings} \end{matrix} \right\}$$

$$(A, I) \longmapsto R = A/I$$

ex:
1) Any perfect $\mathbb{F}_p$-alg S is perfectoid: $S = W(S)/p$
2) $C/\mathbb{Q}_p$ complete + alg closed $\Rightarrow$ $\mathcal{O}_C$ is perfectoid
3) If R is a p-complete regular ring, then $\exists$ f-flat cover $R \to S$ with S perfectoid

   ex: $\mathcal{O}_C[x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty}]^\wedge$

Fix a perfectoid ring R corr to (A, I)

Thm A (André) : Given $g \in R$, $\exists$ a p-comp f flat map $R \to S$ of perfectoids s.t $g^{1/p^\infty} \in S$

More generally: can solve any monic equation this way (above corr to $X - g$)

ex: $R = \mathcal{O}_C[x^{1/p^\infty}]^\wedge$, $g = X - 1$

Proofs: 1) Analytic proof

2) Prismatic proof :

$$S_0 = R[x^{1/p^\infty}]/(x - g) \quad \text{is semiperfectoid}$$

$$S = (\Delta_{S_0/A,\ \text{perf}})/d \quad \text{works!}$$

—

Def: For any R-algebra S, set

$$S_{\text{perfd}} = (\Delta_{S/A,\ \text{perf}})/d \in D(S)$$

Properties: 1) Almost math along closed subsets :

Claim: If $S = R/J$ for some $J \subset R$, then $S_{perfd} \cong R/J_{perfd}$ for $J \subset J_{perfd} \subset R$ and is perfectoid

$$\Rightarrow \quad J_{perfd} \hat{\otimes}^{L} J_{perfd} \cong J_{perfd}$$

$\Rightarrow \exists$ a good notion of almost mathematics wrt J:

$$D_{p\text{-comp}}(R) \Big/ D_{p\text{-comp}}(R/J_{perfd})$$

__Pf sketch__: 1) If $J = (g)$ for $g^{1/p^\infty} \in R$, then $J_{perfd} = (g^{1/p^\infty})^\wedge$ works

2) Use Thm A to reduce to Step 1 $\square$

2) Excision: If $S_1 \to S_2$ is an integral map that is an isom after inv $g \in R$, then

$$(S_1)_{perfd} \xrightarrow{\sim} (S_2)_{perfd}$$

↑ g-almost isom.

(Pf uses arc descent)

II) __Direct summand conjecture__ (Hochster, 69)

__Thm D__ (André, 2016): Say $R_0 \hookrightarrow S_0$ is a finite injective map with R_0 regular

$\Rightarrow R_0 \to S_0$ splits in $\mathrm{Mod}(R_0)$

Rmk: 1) Was known if R_0 contains a field (Hochster) or $\dim \leq 3$ (Heitmann)

2) Has many implications, ex: descent of flatness along integral maps (Ohi, Raynaud-Gruson)

3) Has been generalized substantially

ex: (B, 2020): R_0 is p-complete, then $R_0 \to R_0^+$ (abs int closure) is faithfully flat.

Ref: "Cohen-Macaulayness of absolute integral closures"

Sketch of pf of Thm D via Thm A:

a) Assume R_0 is p-complete and $\exists\ 0 \neq g \in R_0$ s.t. $R_0[\frac{1}{g}] \to S_0[\frac{1}{g}]$ is finite etale Galois with group G and defined by a monic equation

b) Thm A $\Rightarrow \exists$ a faithfully flat cover $R_0 \to R$ with R perfectoid and abs. int closed

$\therefore$ If $S = S_0 \otimes_{R_0} R$, then $R[\frac{1}{g}] \to S[\frac{1}{g}]$

is a totally split Galois cover (i.e. $S[\frac{1}{g}] = \prod_{i=1}^{n} R[\frac{1}{g}]$)

c) Excision: $S_{perfd} \cong \prod_{i=1}^{n} R$

↑ g-almost isom

$\therefore$ $R \to S \to S_{perfd}$ is g-almost split

$\Rightarrow$ $R \to S$ is g-almost split

d) Noetherianess $\Rightarrow$ $R_0 \to S_0$ is actually split. $\square$

III) <u>K-theory</u>:

<u>Motivation</u> (Bott): For a reasonable space X,

$\pi_{odd} K(-)$ is locally zero

<u>Thm K</u> (B. Scholze): On quasi-syntomic rings,

$\pi_{odd} K(- ; \mathbb{Z}_p)$ is locally zero

in the quasi-syntomic top.

Rmk: 1) Thm K was conjectured jointly with Morrow (and also proven in char p there)

2) Pf method $\Rightarrow$ $\pi_{odd} K(\mathcal{O}_{\mathbb{C}_p}/p^n ; \mathbb{Z}_p) = 0$

[Curiously: $\pi_{2i} K(\mathcal{O}_K/p^n) = 0$ $\forall i >> 0$ and $K/\mathbb{Q}_p$ finite ext. (Antieau - Krause - Nikolaus)]

Pf sketch:

1) **Thm** (Clausen-Mathew-Morrow, BMS2)

$\exists$ a filtration on $K_{et}(R; \mathbb{Z}_p)$ with

$$gr_i \cong \mathbb{Z}_p(i)(R)[2i]$$

$$\text{fib}\left(Fil^i_N \Delta_R\{i\} \xrightarrow{\phi_i - 1} \Delta_R\{i\}\right)[2i]$$

Thm K $\Leftarrow$ $\mathbb{Z}_p(i)(-)$ is quasi-syn locally in deg 0

[**Rmk:** $\mathbb{Z}_p(0) = \mathbb{Z}_p$, $\mathbb{Z}_p(1) = T_p \mathbb{G}_m$

In char p, X smooth, $\mathbb{Z}_p(i)|_{X_{et}} = W\Omega^i_{log}[-i]$]

2) Check $\mathbb{Z}_p(i)(R)$ is in deg 0

for $R = \left(\mathcal{O}_C[x_1^{1/p^\infty}, \ldots, x_n^{1/p^\infty}]^\wedge\right) / (x_1, \ldots, x_n)$

by hand

3) Reduce to the case in (2) using Thm A

Key pt: R perfectoid, $g^{1/p^\infty} \in R$

$$\Rightarrow \frac{R[x^{1/p^\infty}]}{(x)} \xrightarrow{x^{1/p^n} \mapsto g^{1/p^n}} R/g$$

is surjective on $\Delta_{(-)}$