The Hecke Orbit conjecture

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Résumé

We report on joint work with Ching-Li Chai.
The *isogeny class* of a polarized abelian variety (A,μ) with moduli point $x = [(A,\mu)]$ is denoted by $H(x)$, called the *Hecke orbit* of x. What is the Zariski closure of the set $H(x)$? In characteristic zero any Hecke orbit is everywhere dense in the moduli scheme of polarized abelian varieties. We easily see that this is not true in general (in the non-ordinary case) in positive characteristic. We consider and prove:

Conjecture / Theorem **(HO).** *The Hecke orbit* $H([(A,\mu)])$ *is everywhere dense in the NP* stratum given by $\xi = N(A)$.

This was proved by Ching-Li Chai for *ordinary* abelian varieties (Invent. Math. 1995) and formulated as a conjecture in the general case (FO, 1995).

In order to obtain a proof, as a starting point, properties of (moduli of) *supersingular abelian varieties* were established. Several (new) concepts and their properties were needed:

- *stratifications* (EO and NP),
- *foliations* (central leaves, isogeny leaves),
- minimal p-divisible groups, hypersymmetric abelian varieties,
- a proof of a conjecture by Grothendieck,
- a generalization of Serre-Tate coordinates,

in order to understand the *geometry* of these moduli spaces in positive characteristic. Many aspects of these had to be established, such as dimension, irreducibility, the graph of strata with partial ordering given by "inclusion in the boundary". This might explain why it took us 25 years to prove the conjecture.

In this talk I will explain these notions and formulate the *basic strategy of approach*: specialization to the "boundary" of various strata. Examples and relevant structure theorems will be amply discussed. Then I will sketch a proof of (HO), divided into two aspects: the discrete and the continuous HO problem.

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